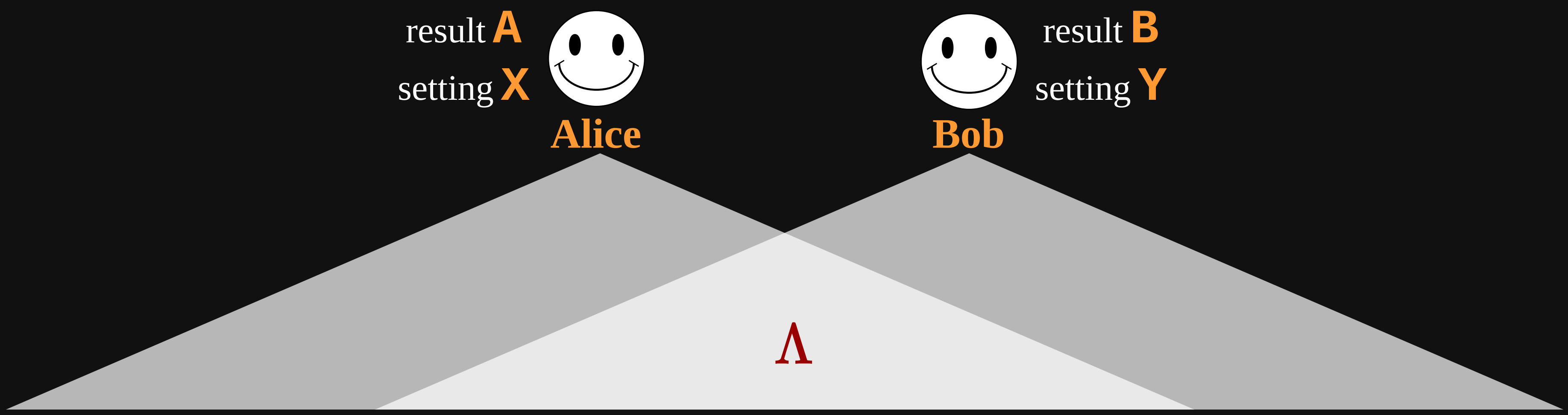
Möbius etc.

Relativistic and Logical Bounds on Causality

Ämin Baumeler, Università della Svizzera italiana

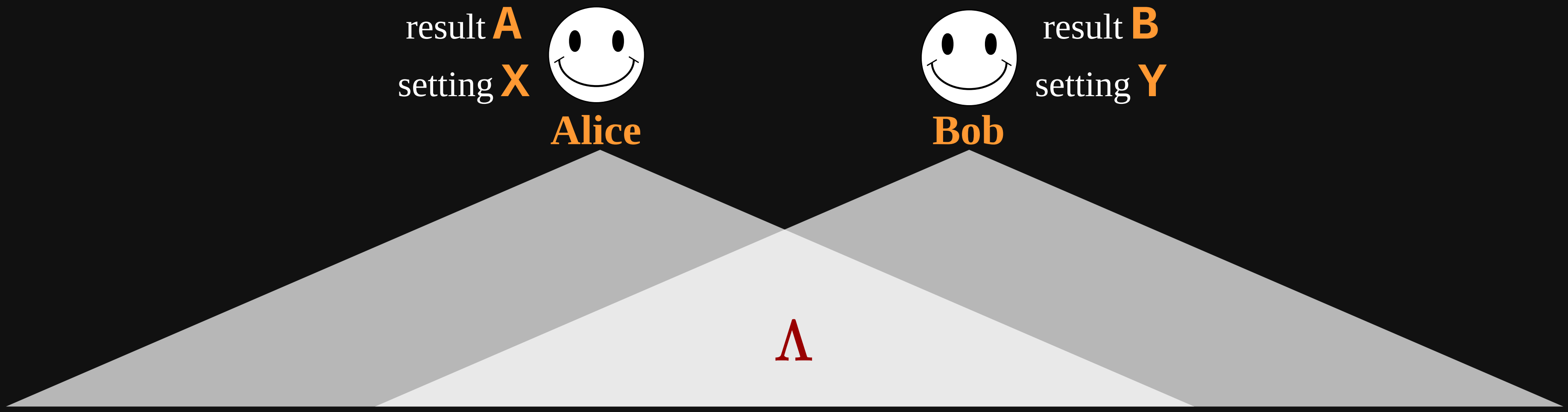
November 13, 2024 @ **SWIT**

John Bell



$$p(a,b|x,y) = \sum_{\lambda} p(\lambda)\ \ p(a|x,\lambda)\ \ p(b|y,\lambda)$$

John Bell



$$p(a,b|x,y) = \sum_{\lambda} p(\lambda)\ p(a|x,\lambda)\ p(b|y,\lambda)$$

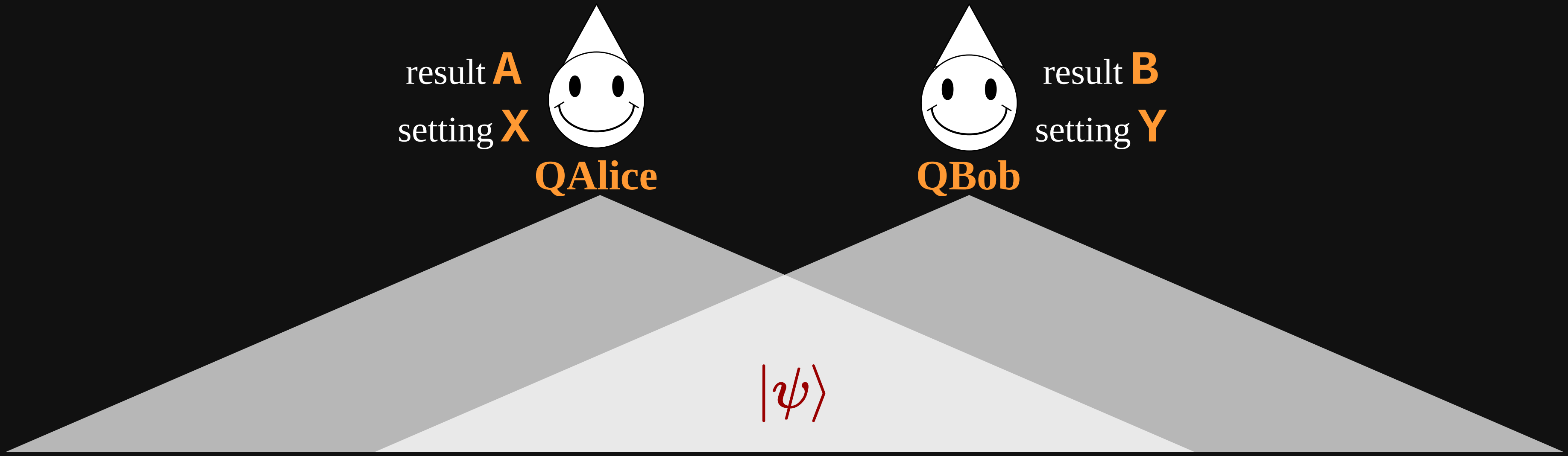
The Bell game

X	Y	
0	0	$A = B$
0	1	$A = B$
1	0	$A = B$
1	1	$A \neq B$

$$\Pr[A \oplus B = X \cdot Y] \leq 3/4$$

for uniformly distributed X, Y

Quantum Bell



$$\Pr_{Q}[A \oplus B = X \cdot Y] = \frac{2+\sqrt{2}}{4}$$

Violation of Bell's inequality

Drop *at least one of*:

1. Causal structure
2. Parameter independence
3. Locality

Violation of Bell's inequality

Drop *at least one of*:

1. Causal structure
2. Parameter independence
3. ~~Locality~~

Far-reaching consequences

- *Device-independent* cryptography
- Computational advantage

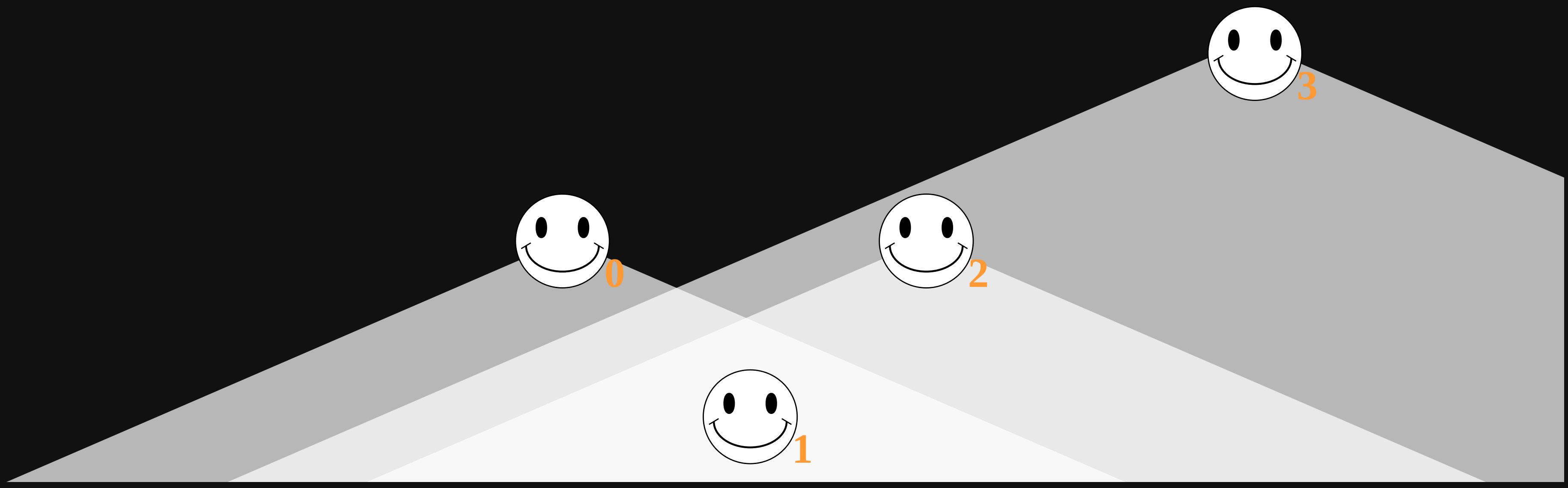
Outline

1. Static causal order *(special relativity)*
2. Dynamic causal order *(general relativity)*
3. No causal order

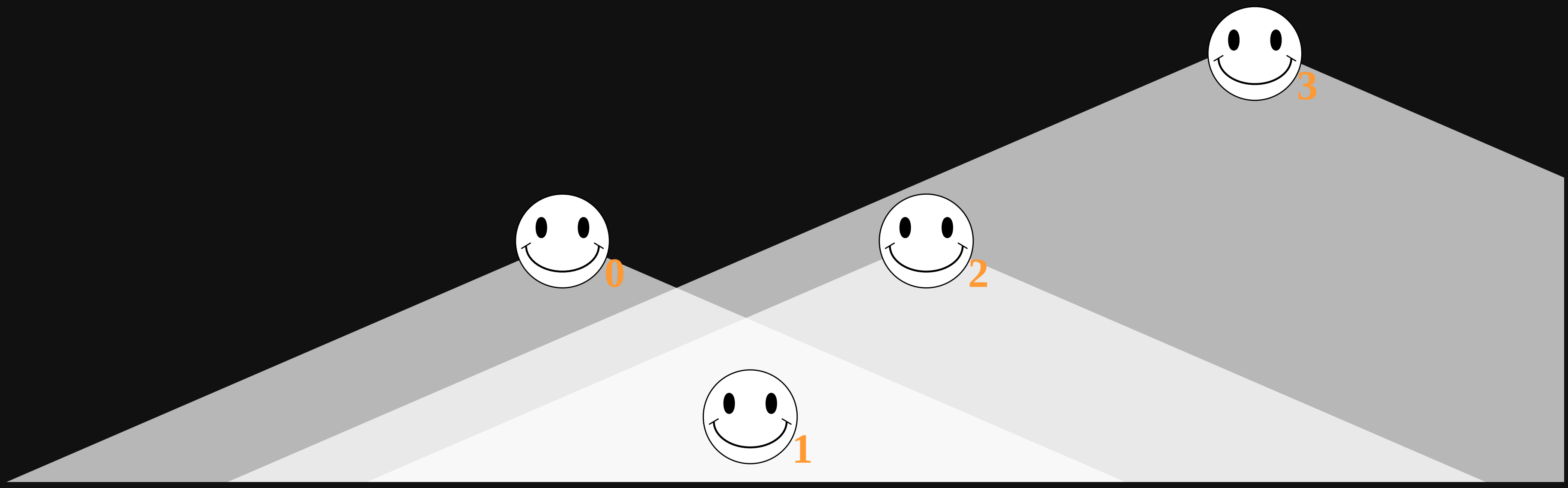
Outline

1. Static causal order *(special relativity)*
2. Dynamic causal order *(general relativity)*
3. No causal order

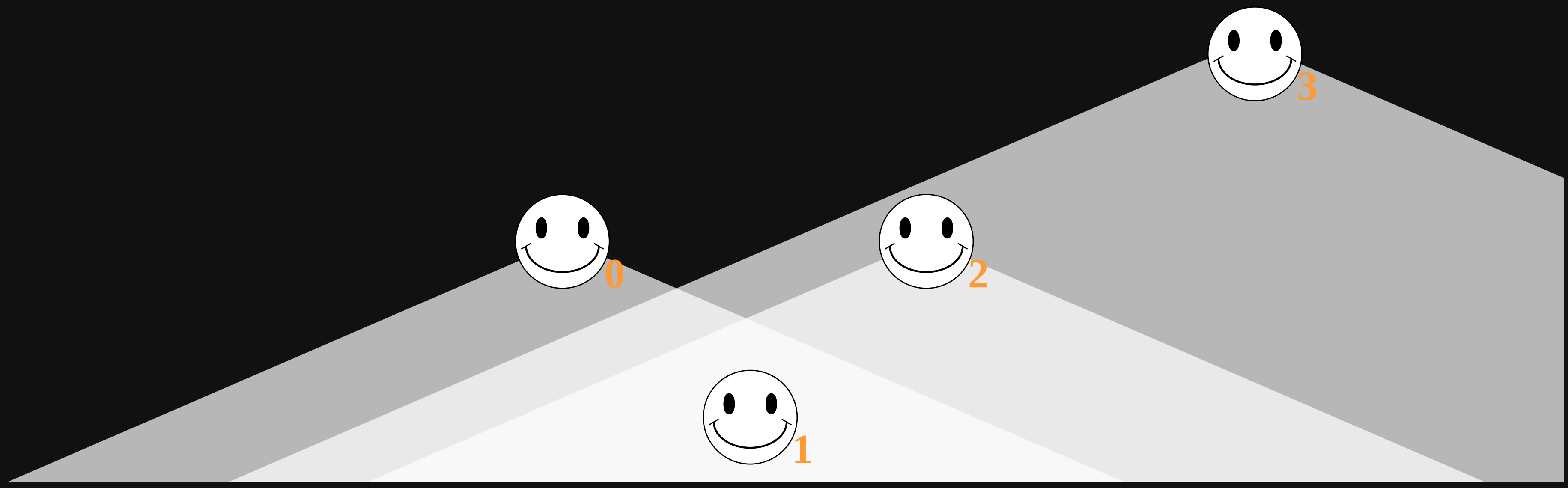
Static causal order



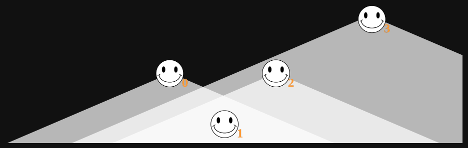
Observations at *fixed* spacetime coordinates



Observations obey *partial order*



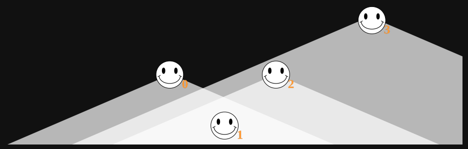
- Result A_3 may depend on setting X_1
- Result A_1 *cannot* depend on setting X_3



Decomposition of

$$p(a_0, a_1, a_2, \ldots | x_0, x_1, x_2, \ldots)$$

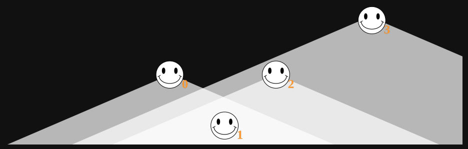
$$p(\underline{a}|\underline{x}) = \sum_{\sigma} p(\sigma) \prod_{k} p(a_k|\sigma, x_k, a_{\prec_\sigma k}, x_{\prec_\sigma k})$$



Decomposition of

$$p(a_0, a_1, a_2, \ldots | x_0, x_1, x_2, \ldots)$$

$$p(\underline{a}|\underline{x}) = \sum_{\sigma} p(\sigma) \prod_{k} p(a_k | \sigma, x_k, a_{\prec_\sigma k}, x_{\prec_\sigma k})$$

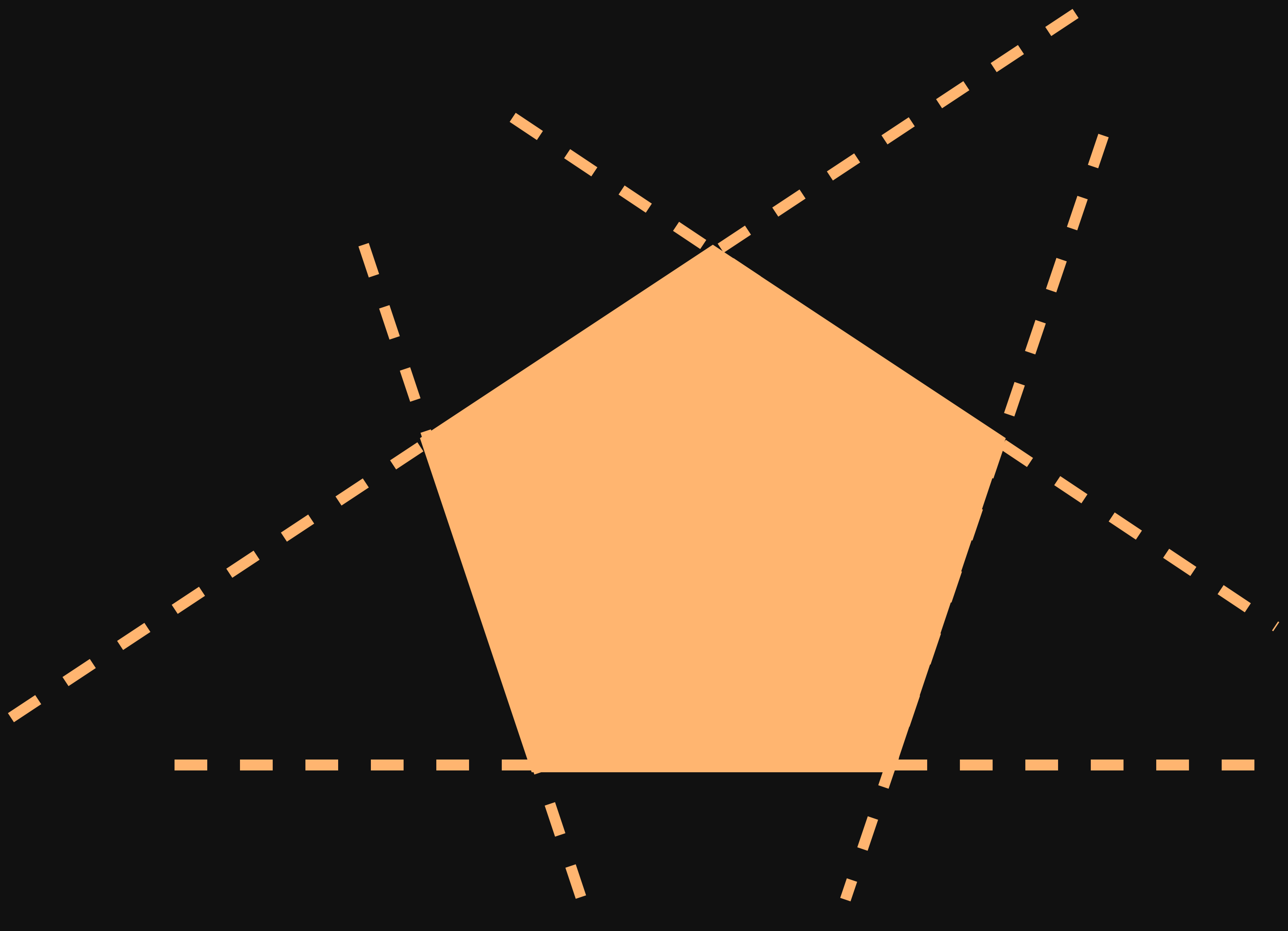


Decomposition of

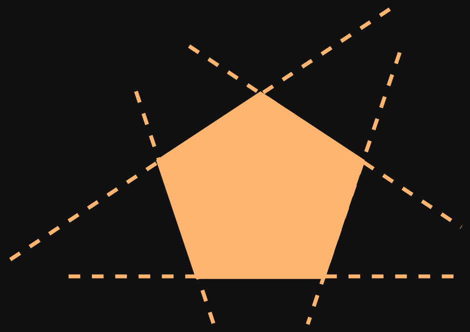
$$p(a_0, a_1, a_2, \dots | x_0, x_1, x_2, \dots)$$

$$p(\underline{a}|\underline{x}) = \sum_{\sigma} p(\sigma) \prod_{k} p(a_k|\sigma, x_k, a_{\prec_\sigma k}, x_{\prec_\sigma k})$$

Polytope



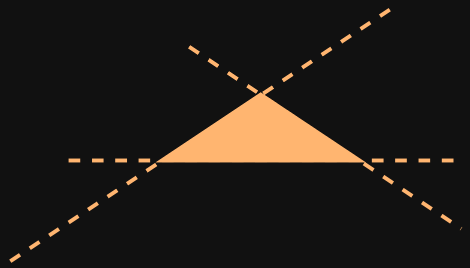
Polytope: properties



- $2n(n-1)$-dimensional
- $0/1$
- Pairwise central symmetry

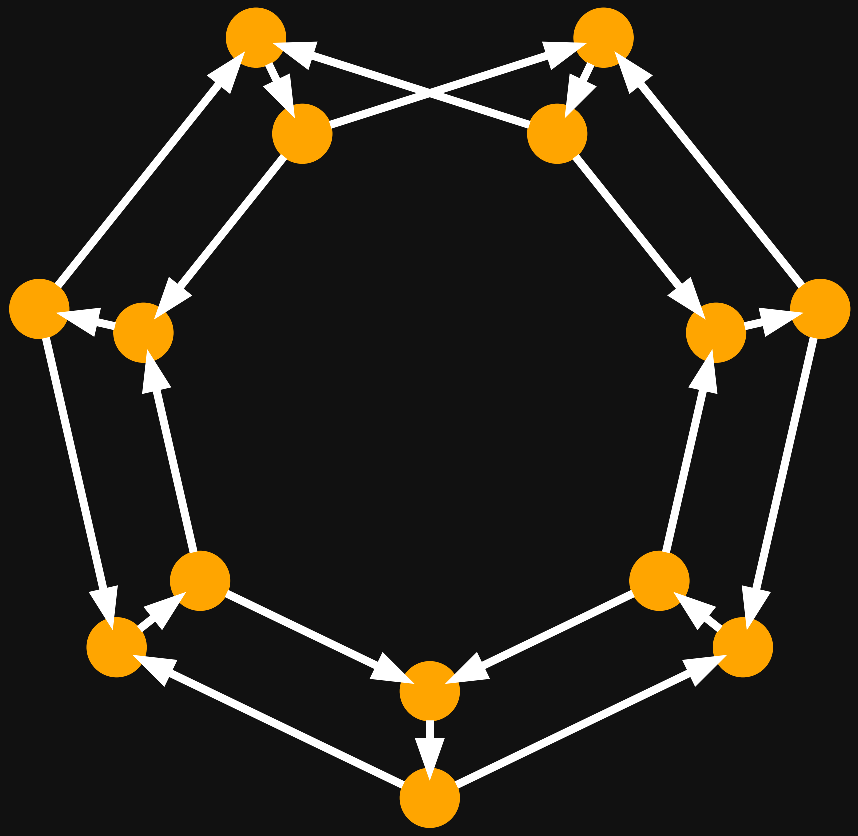
$\Rightarrow$ Project & lift

DAG polytope

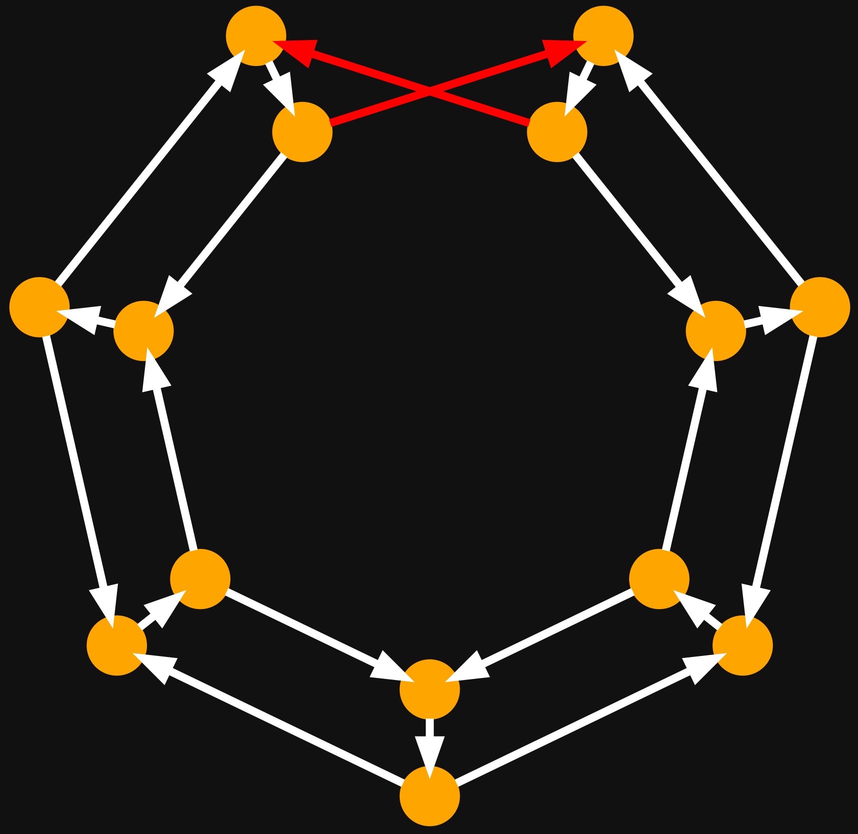


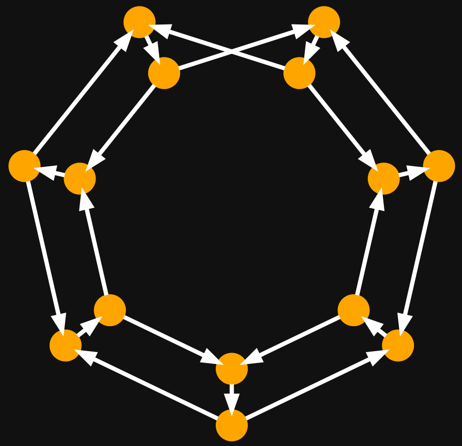
- $n(n-1)$ dimensional
- $0/1$
- Extremal points are **D**irected **A**cyclic **G**raphs

The Möbius game

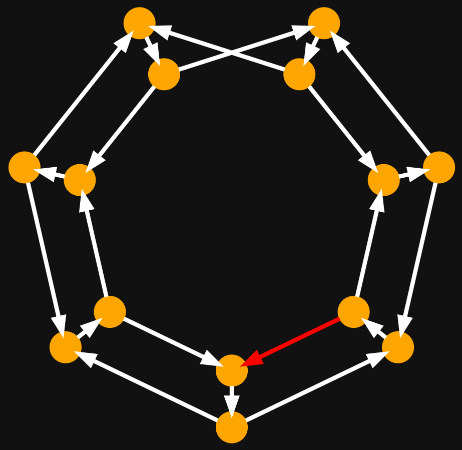


The Möbius game

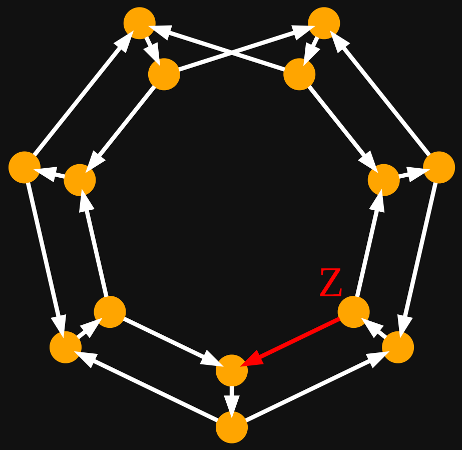




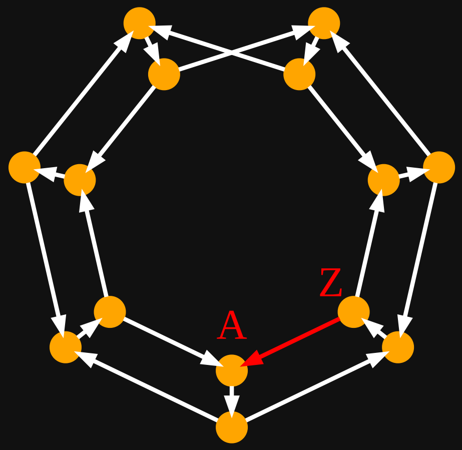
1. Referee selects random arc $(s \longrightarrow r)$
2. Referee gives random bit Z to *sender* s
3. *Receiver* r must output $A = Z$



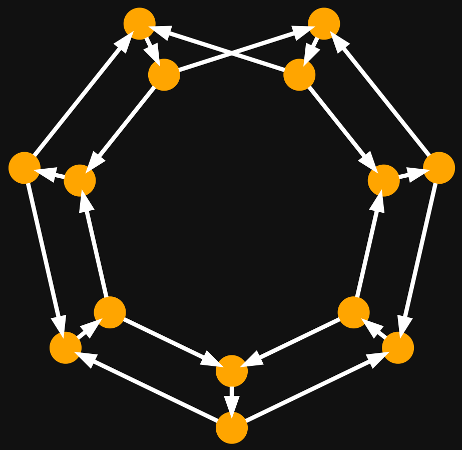
1. Referee selects random arc $(s \longrightarrow r)$
2. Referee gives random bit Z to *sender* s
3. *Receiver* r must output $A = Z$



1. Referee selects random arc $(s \longrightarrow r)$
2. Referee gives random bit Z to *sender* s
3. *Receiver* r must output $A = Z$



1. Referee selects random arc $(s \longrightarrow r)$
2. Referee gives random bit Z to *sender* s
3. *Receiver* r must output $A = Z$



$$\Pr[A = Z] \leq 1 - \frac{k+1}{12k} \leq \frac{11}{12}$$

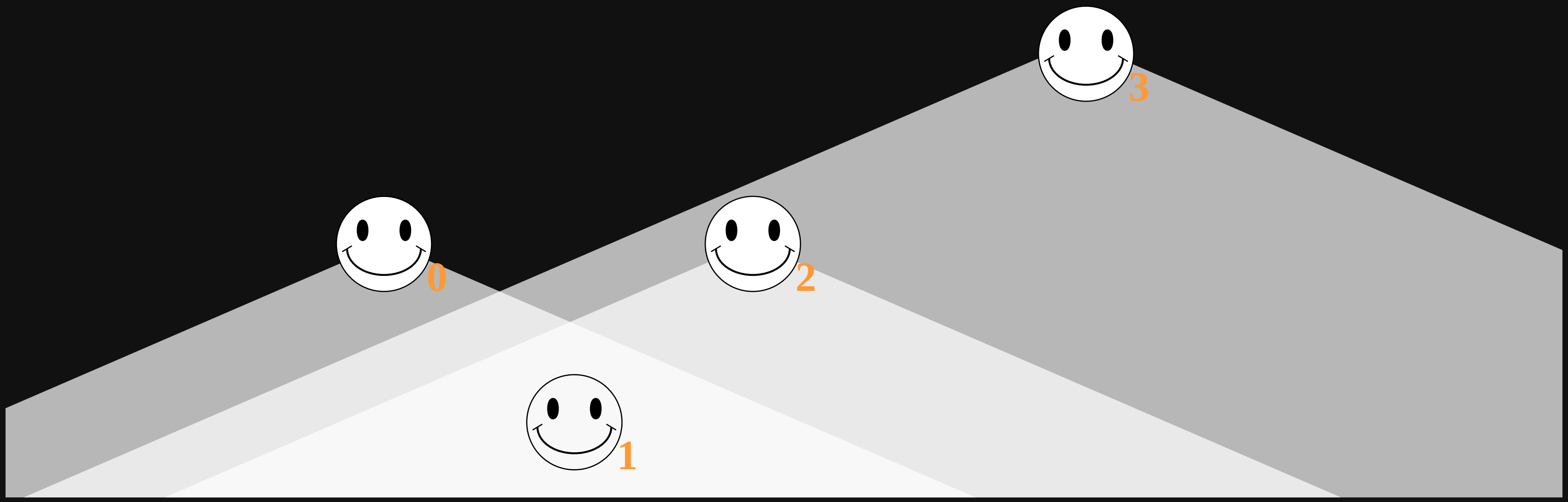
Violation of the Möbius inequality …

… proves *incompatibility* with partial order.

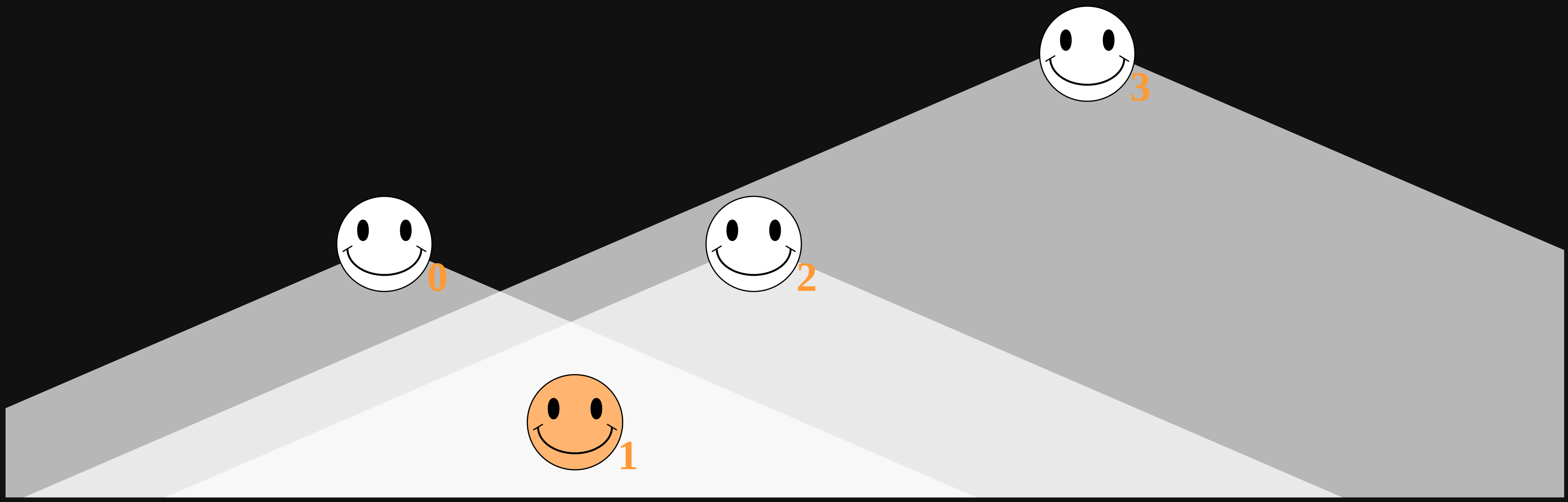
Violation of the Möbius inequality …

… proves *incompatibility* with **special relativity.**

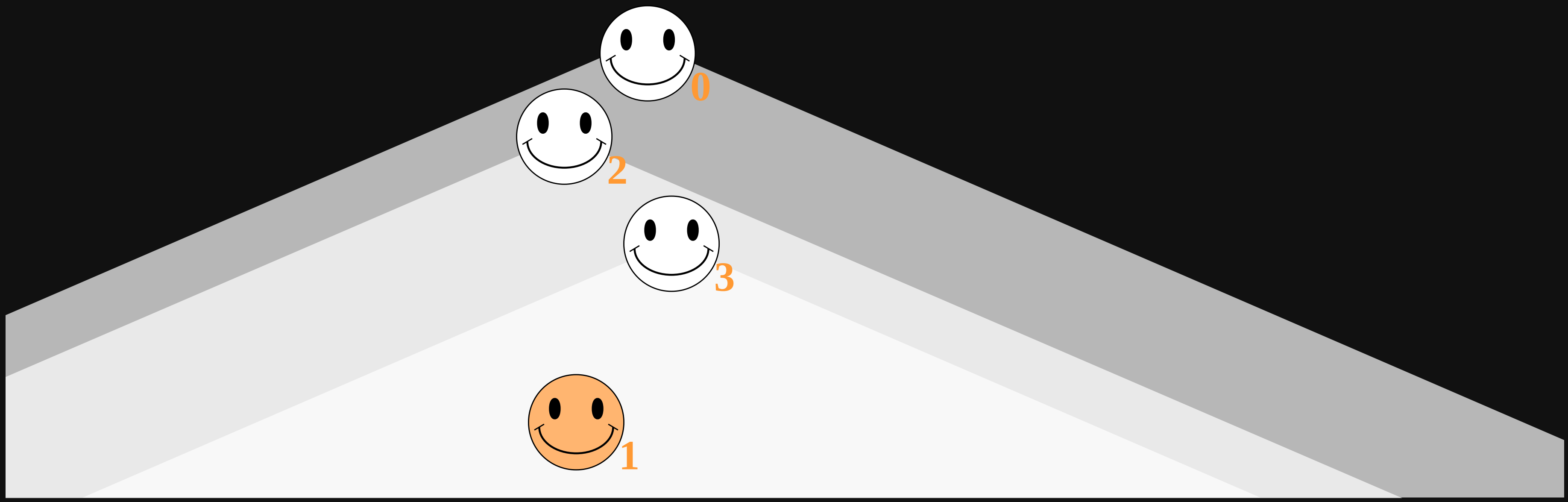
Dynamic causal order



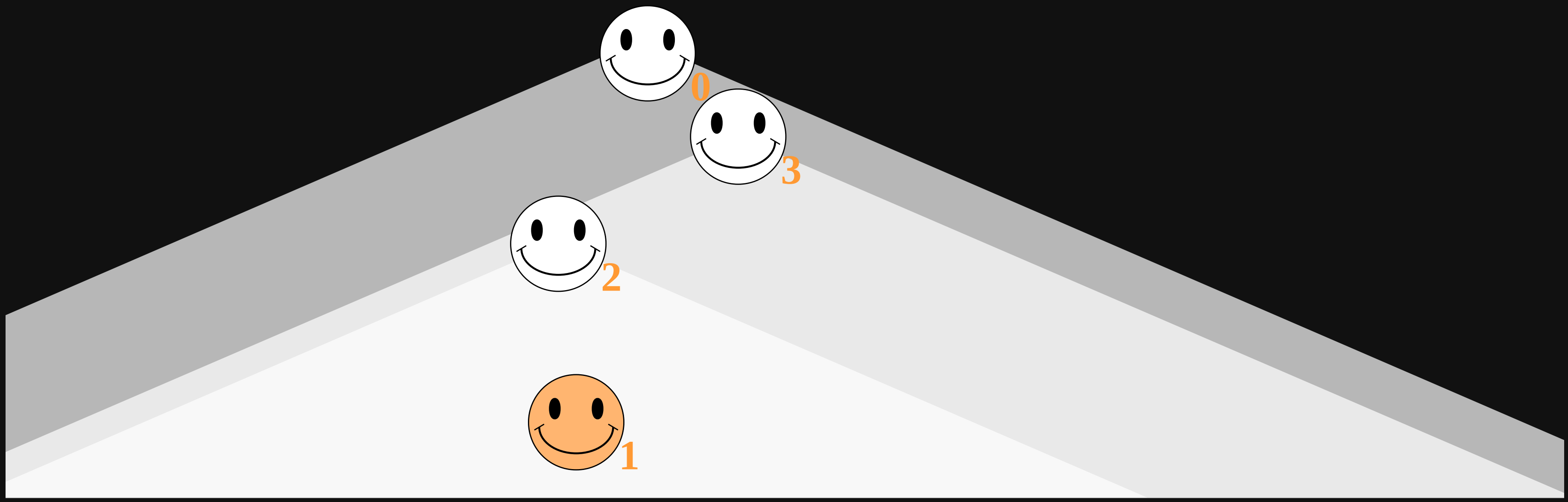
Dynamic: Order of events depends on past



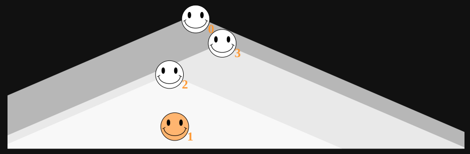
Dynamic: Order of events depends on past



Dynamic: Order of events depends on past



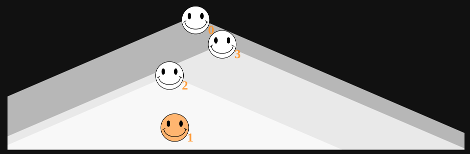
Dynamic: Order of events depends on past



Recursive decomposition of $p(\underline{a}|\underline{x})$:

$$p(\underline{a}|\underline{x}) = \sum_k p(k)\ \ p(a_k|x_k)\ \ p_k(\underline{a}_{\setminus k}|\underline{x}_{\setminus k})$$

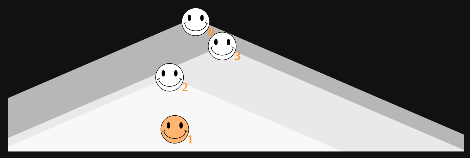
where $p_k(\underline{a}_{\setminus k}|\underline{x}_{\setminus k})$ decompose in the same way.



Recursive decomposition of $p(\underline{a}|\underline{x})$:

$$p(\underline{a}|\underline{x}) = \sum_k p(k)\ \ p(a_k|x_k)\ \ p_k(\underline{a}_{\setminus k}|\underline{x}_{\setminus k})$$

where $p_k(\underline{a}_{\setminus k}|\underline{x}_{\setminus k})$ decompose in the same way.

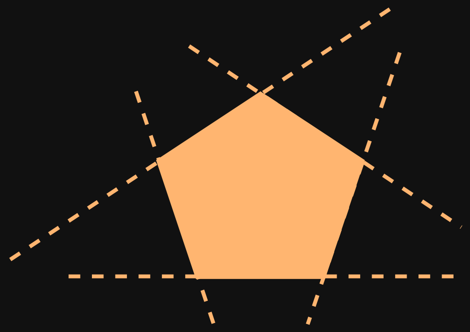


Recursive decomposition of $p(\underline{a}|\underline{x})$:

$$p(\underline{a}|\underline{x}) = \sum_k p(k) \;\; p(a_k|x_k) \;\; p_k(\underline{a}_{\setminus k}|\underline{x}_{\setminus k})$$

where $p_k(\underline{a}_{\setminus k}|\underline{x}_{\setminus k})$ decompose in the same way.

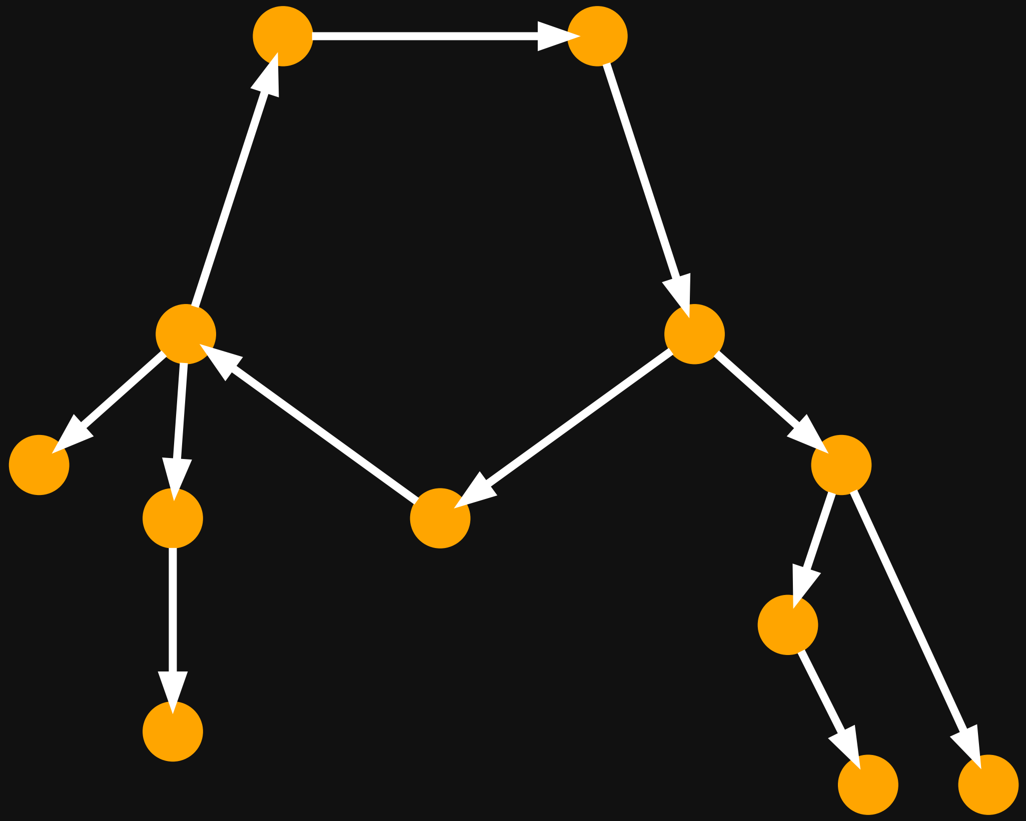
Polytope: same properties



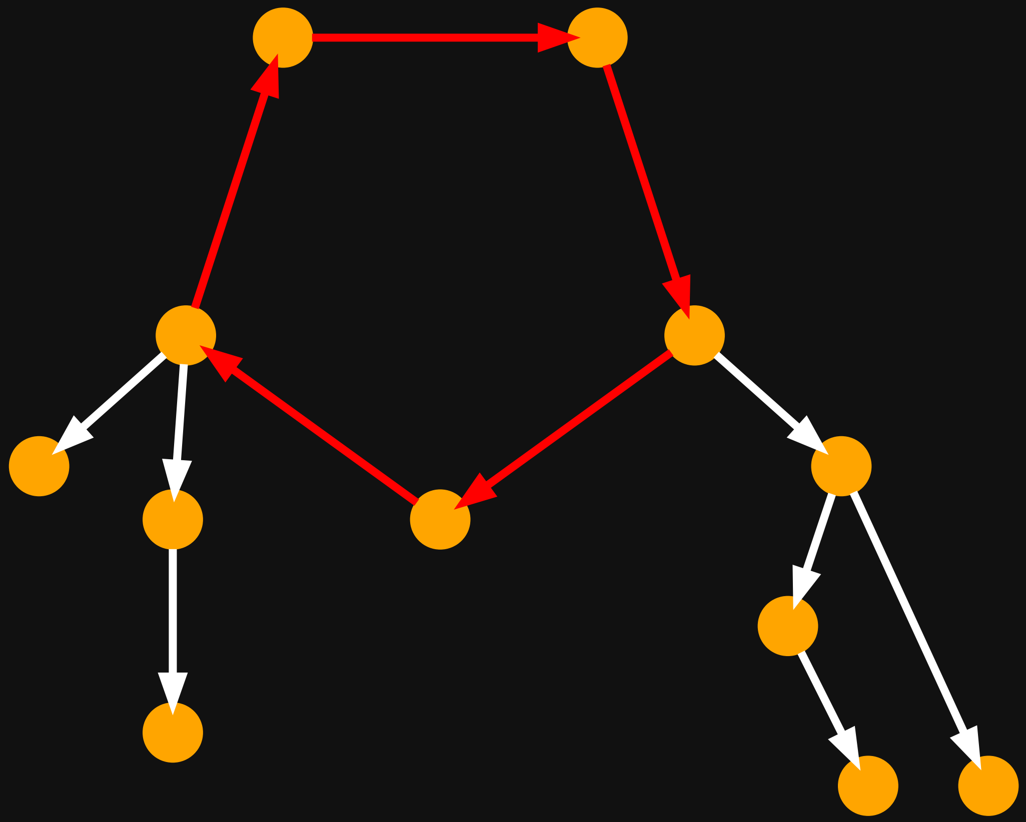
- $2n(n-1)$-dimensional
- $0/1$
- Pairwise central symmetry

$\Rightarrow$ Project & lift

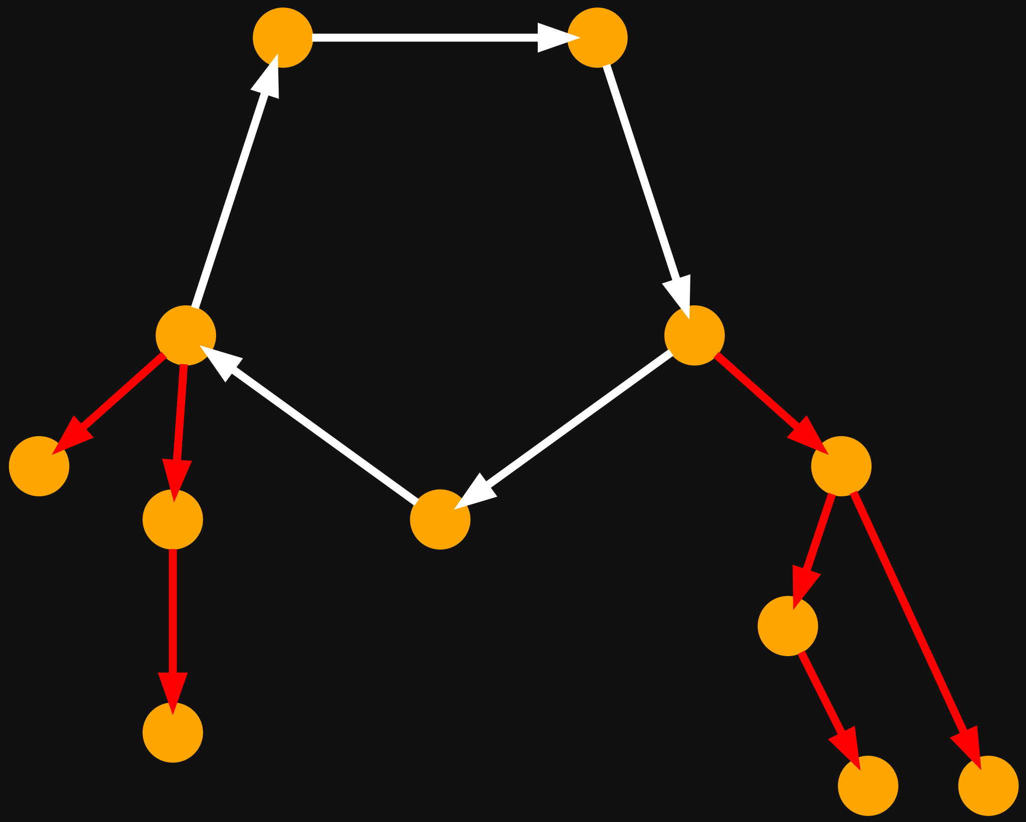
The Cephalopoda game

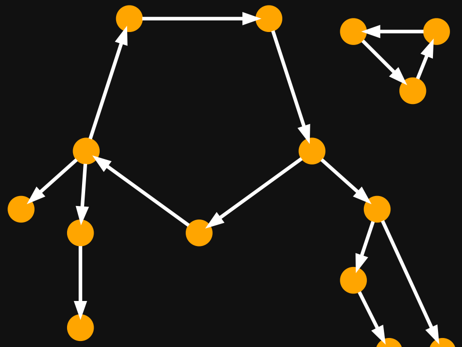


The Cephalopoda game



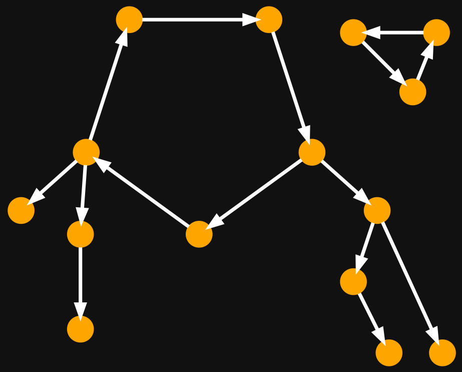
The Cephalo**poda** game





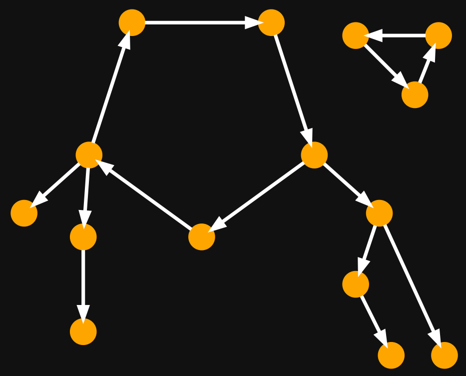
Cephalopod: A directed cycle (head) with any number of attached out-trees (feet)

Cephalopoda digraph: Any number of disconnected cephalopods

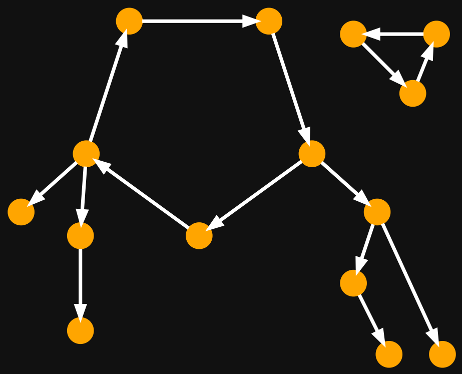


Alternative definition: $\forall v : deg_{\text{in}}(v) = 1$

These graphs on n vertices have *exactly* n arcs.



1. Referee selects random arc $(s \longrightarrow r)$
2. Referee gives random bit Z to *sender* s
3. *Receiver* r must output $A = Z$



$$\Pr[A = Z] \leq 1 - \frac{1}{2n}$$

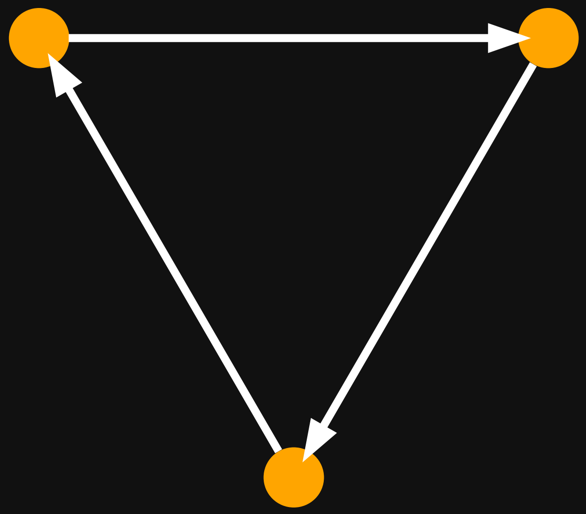
Violation of the Cephalopoda inequality …

… proves *incompatibility* with dynamical causal order.

Violation of the Cephalopoda inequality ...

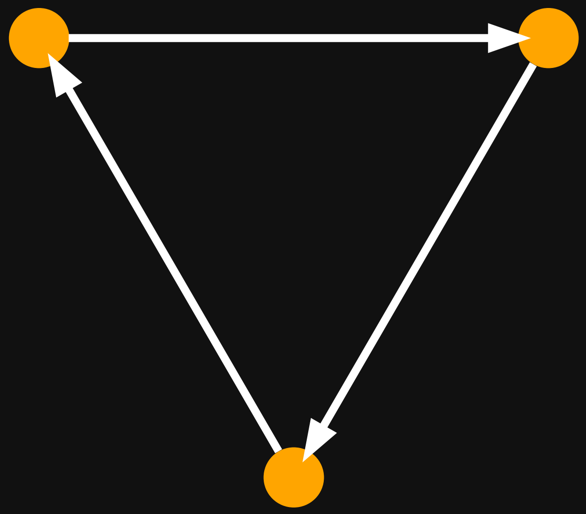
... proves an *incompatibility* with general relativity?

No causal order



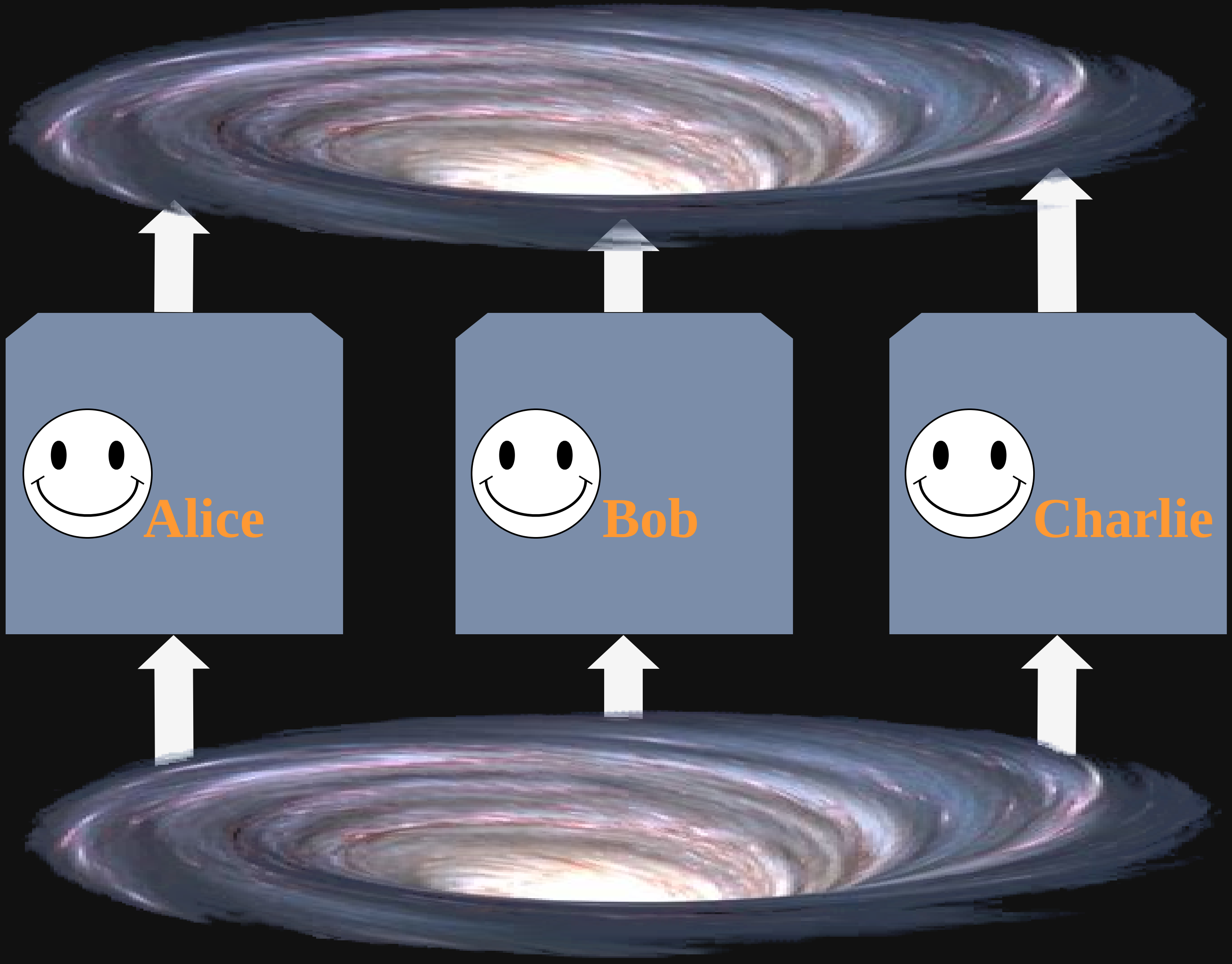
$$\Pr[A = Z] \leq 5/6$$

Violation possible?

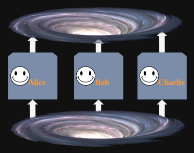


$$\Pr[A = Z] \leq 2/3$$

Violation **logically** possible?

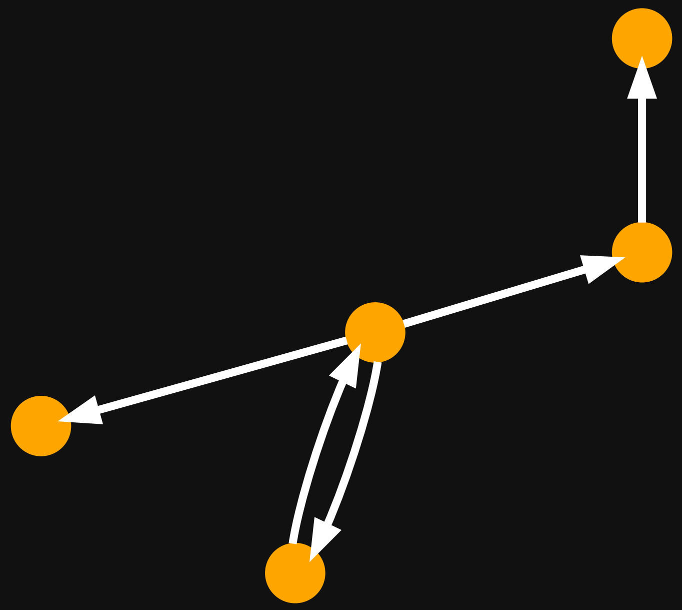

Alice
Bob
Charlie

(Quantum) causal models

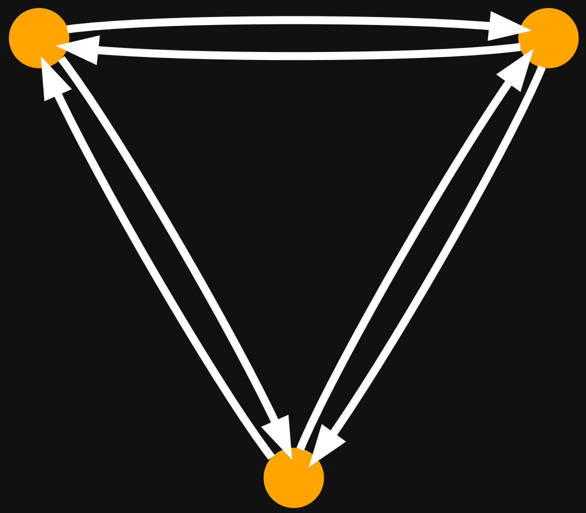


- No assumption on causal order
- Parties perform arbitrary operations
- Well-defined behavior

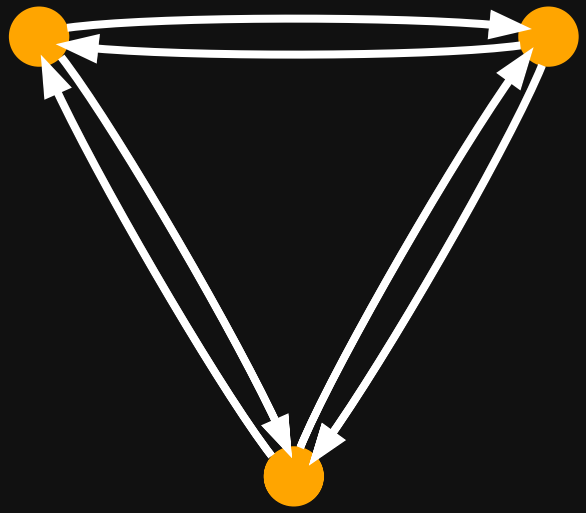
(Quantum) causal models



Example

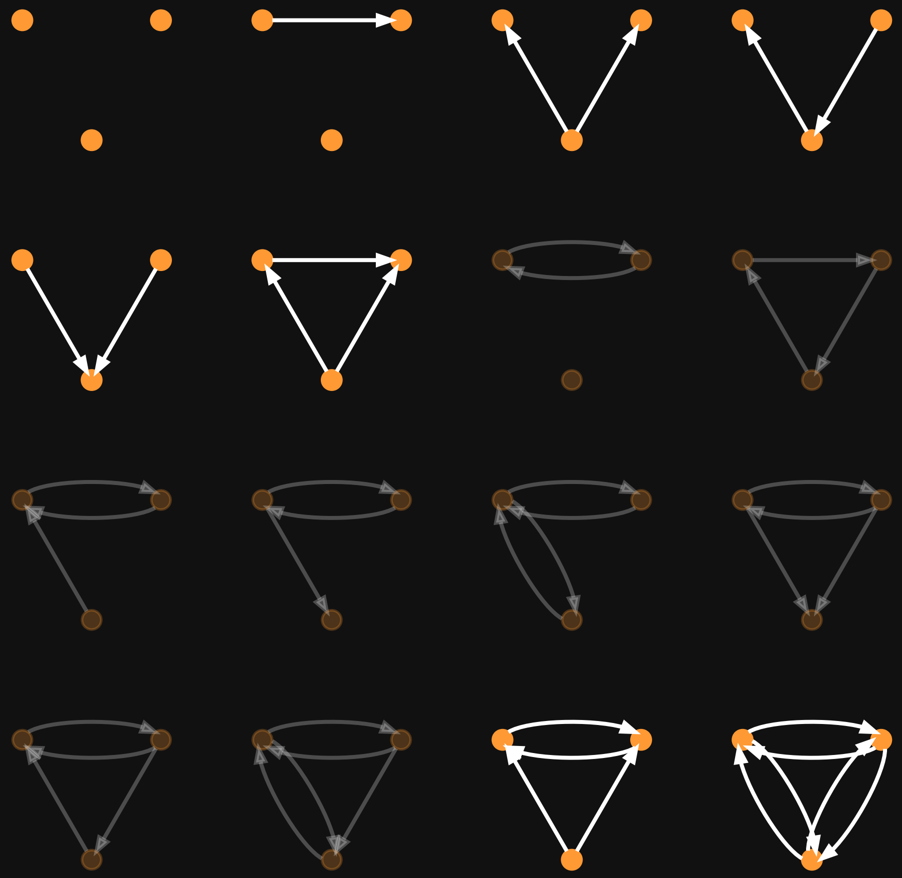


Example



$$\Pr[A = Z] = 1$$

Siblings on cycles



Summary

1. Möbius graphs bound *static causal order* **(SR)**

   Tselentis & Baumeler (2023). The Möbius game and other Bell tests for relativity. arXiv:2309.15752.

   Tselentis & Baumeler (2024). The Möbius game: a quantum-inspired test of general relativity. arXiv:2407.17203.

2. Cephalopoda graphs bound *dynamic causal order* **(GR)**

3. Sibling-on-cycles graphs bound processes *without causal order*

   Tselentis & Baumeler (2023). Admissible causal structures and correlations. PRX Quantum, 4(4), 040307.

Thank You

References

- Möbius (1886). Gesammelte Werke (Zweiter Band). S. Hirzel.
- Bell (1964). On the Einstein Podolsky Rosen paradox. Physics Physique Fizika, 1(3), 195–200.
- Grötschel, Jünger & Reinelt (1985). On the acyclic subgraph polytope. Mathematical Programming, 33(1), 28–42.
- Colbeck (2006). Quantum and relativistic protocols for secure multi-party computation Ph.D. thesis, University of Cambridge.
- Oreshkov, Costa, & Brukner (2012). Quantum correlations with no causal order. Nature Communications, 3(1), 1092.
- Baumeler & Wolf (2016). The space of logically consistent classical processes without causal order. New Journal of Physics, 18(1), 013036.
- Oreshkov & Giarmatzi (2016). Causal and causally separable processes. New Journal of Physics, 18(9), 093020.
- Bravyi, Gosset, & König (2018). Quantum advantage with shallow circuits. Science, 362(6412), 308–311.
- Barrett, Lorenz, & Oreshkov (2021). Cyclic quantum causal models. Nature Communications, 12(1), 885.
- Tselentis & Baumeler (2023). Admissible causal structures and correlations. PRX Quantum, 4(4), 040307.
- Tselentis & Baumeler (2023). The Möbius game and other Bell tests for relativity. arXiv:2309.15752.
- Tselentis & Baumeler (2024). The Möbius game: a quantum-inspired test of general relativity. arXiv:2407.17203.

Möbius etc.

Relativistic and Logical Bounds on Causality

Ämin Baumeler, Universitá della Svizzera italiana

November 13, 2024 @ SWIT