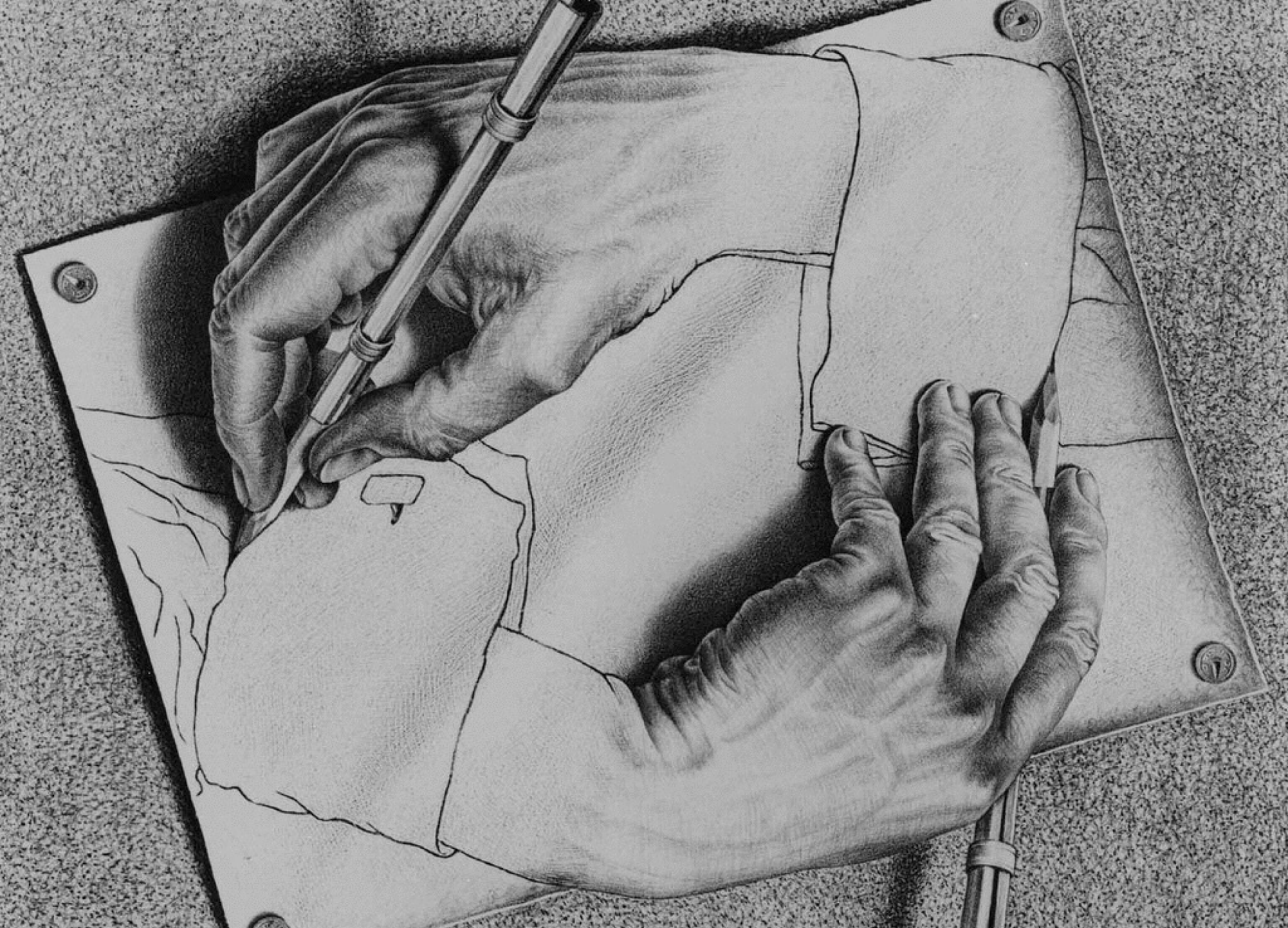
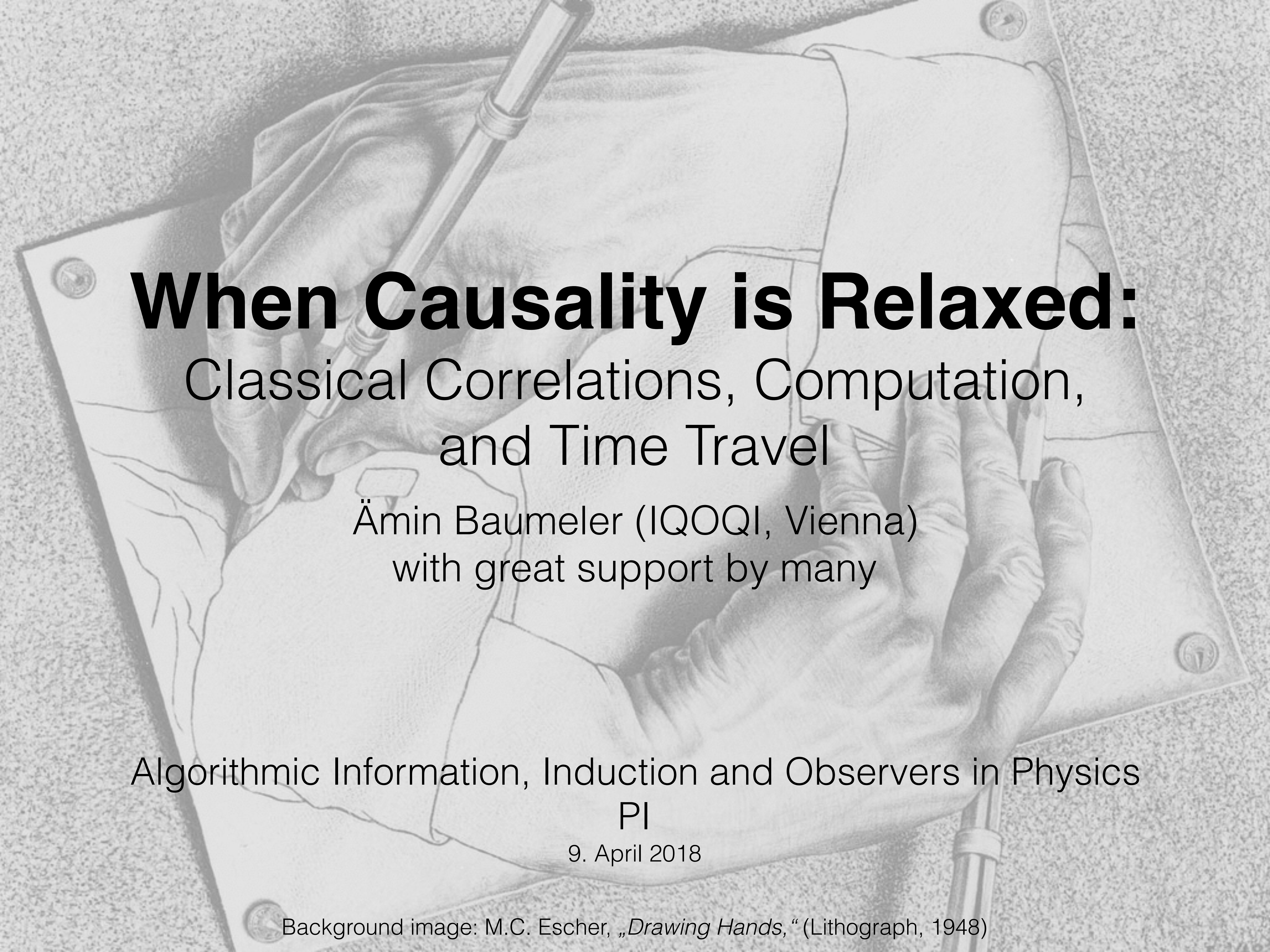


Image: M.C. Escher, „*Drawing Hands,*“ (Lithograph, 1948)



When Causality is Relaxed:

Classical Correlations, Computation, and Time Travel

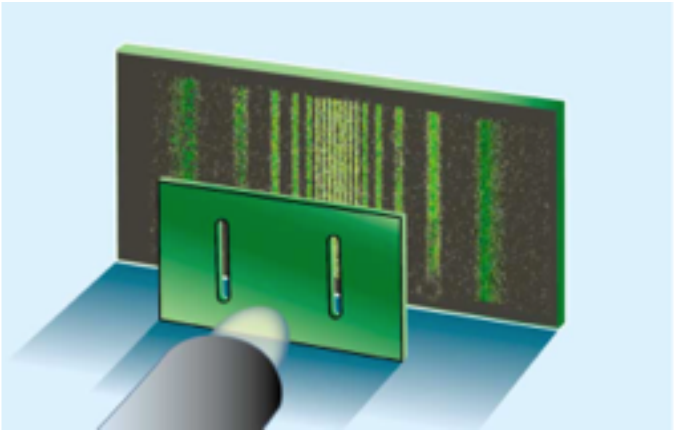
Ämin Baumeler (IQOQI, Vienna)
with great support by many

Algorithmic Information, Induction and Observers in Physics
PI
9. April 2018

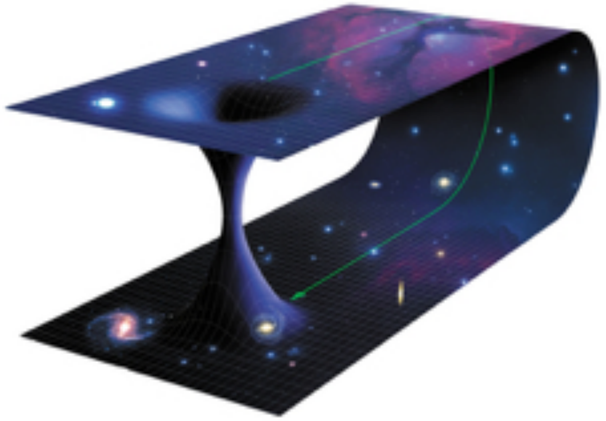
Background image: M.C. Escher, *„Drawing Hands,“* (Lithograph, 1948)


Explaining Correlations in a Causal Structure
The law of causality is a relic of a bygone age, surviving, like the monarchy, only because it is erroneously supposed to do no harm
Bertrand Russell
1913
Drop the Causal Structure?
1956
Hans Reichenbach

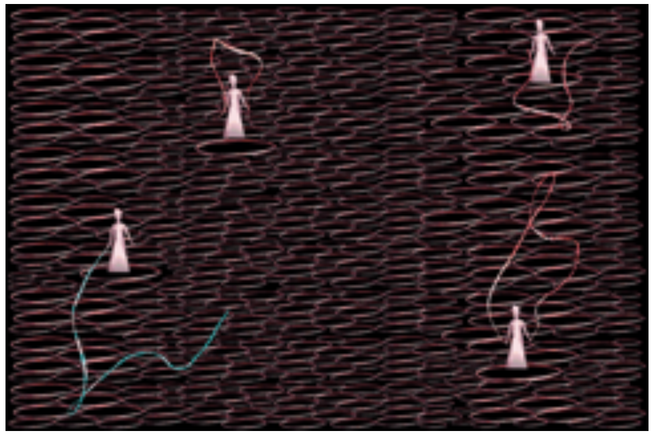
Motivations for Relaxing Causality



(1) Quantum theory
Quantum superpositions
Bell correlations



(2) Relativity theory
Closed time-like curves (CTCs)
e.g., Lanczos (1924), Gödel (1949), Thorne (1988)



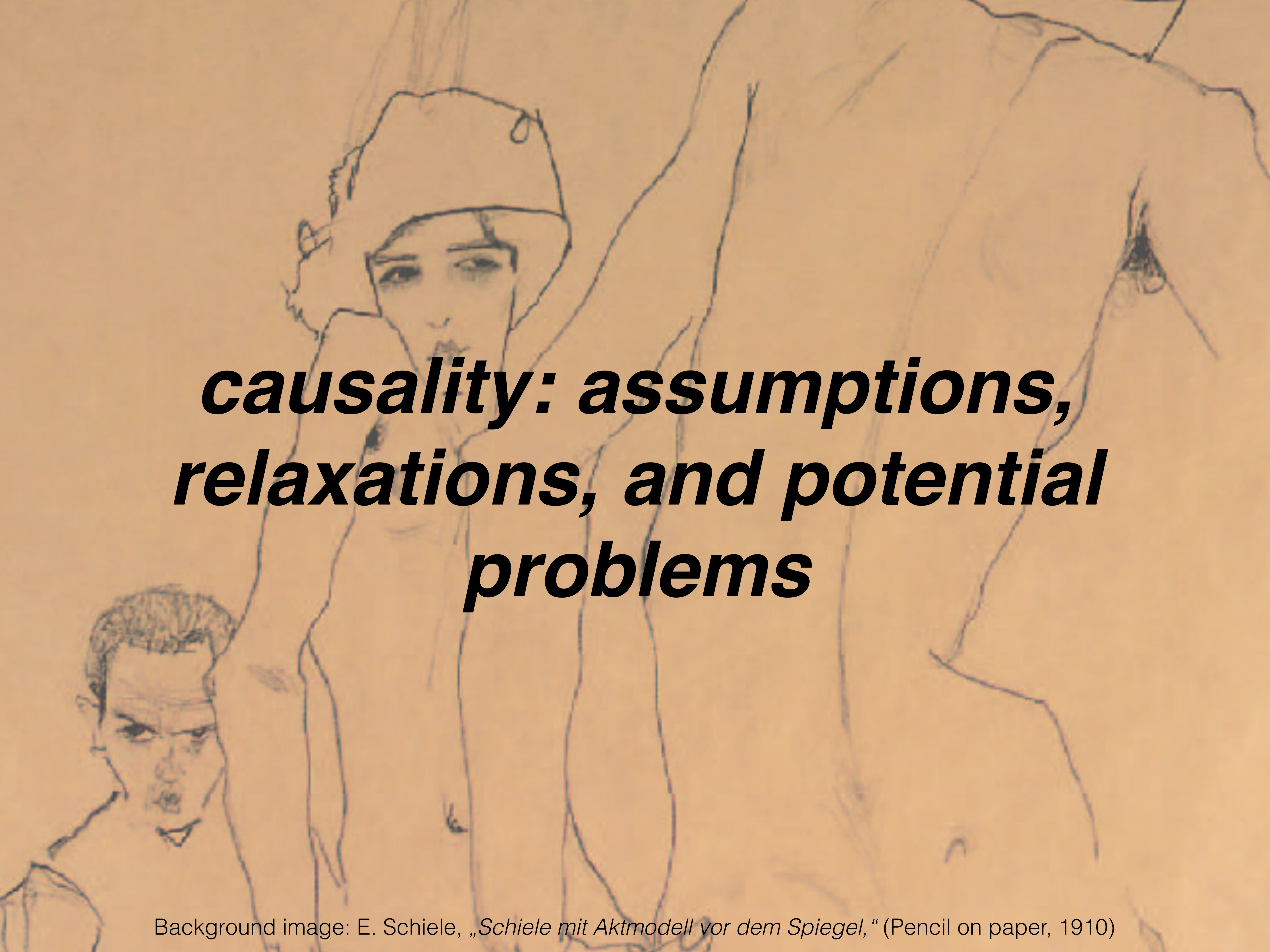
(3) Quantum gravity
GR: dynamic causal structure & deterministic
QT: fixed causal structure & probabilistic

L. Hardy, *arXiv:0509120 [gr-qc]* (2005);
Images: A. Albrecht, *Nature* **412**, 687 (2011); A. Jaffe, *Nature* **537**, 616 (2016); A. Ashtekar, *Nature Physics* **2**, 725 (2006)

Outline

- Motivations
- Causality: assumptions, relaxations, and potential problems
- Classical non-causal correlations
- Non-causal computation
- Time travel
- Conclusion

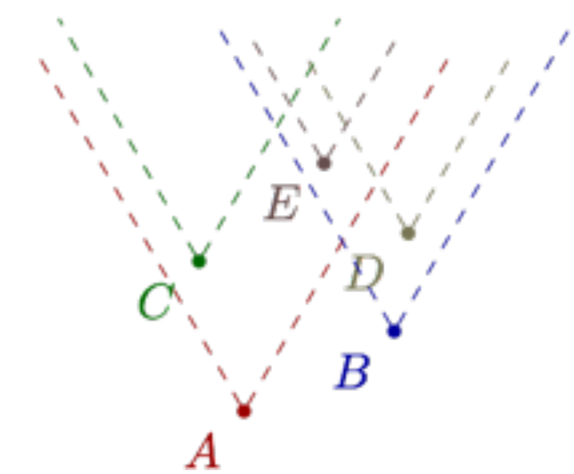
causality: assumptions, relaxations, and potential problems



Background image: E. Schiele, *„Schiele mit Aktmodell vor dem Spiegel,“* (Pencil on paper, 1910)

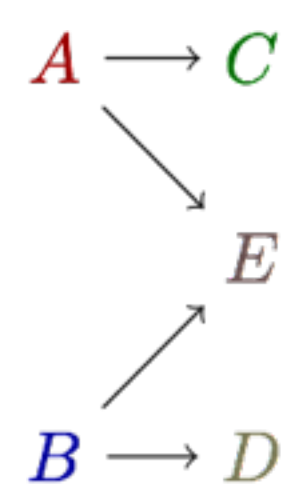
Causal Structures

- *Cause*-*effect* relations
  When I click on this little button *(cause)* you will see the next slide *(effect)*



- Relativity theory: light-cone structure*

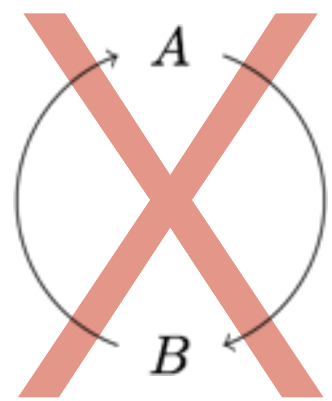
- Modeled as *directed acyclic* graph



* with *postulated* arrow of time

- Traditionally: *definite partial ordering* of events
  Based on intuition, observations; we are used to that

- Partial ordering: no causal loops
  An effect cannot be the cause of the effect's cause (antisymmetric)



- Definite: predetermined, independent of observer
  Fixed causal relations, *e.g.*, no quantum superpositions

$$\frac{1}{\sqrt{2}}|\text{A before B}\rangle + \frac{1}{\sqrt{2}}|\text{A after B}\rangle$$

Relaxing Causality

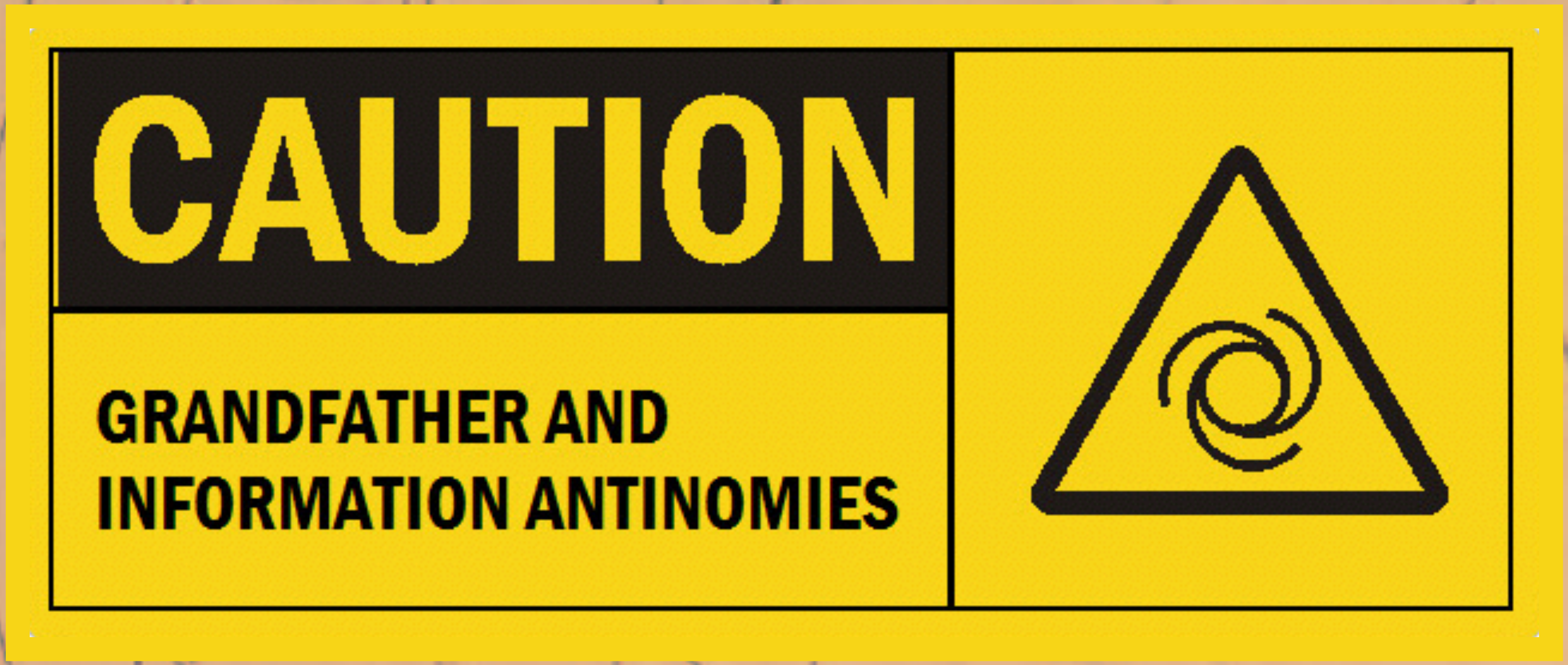
- Drop assumption: *definite partial ordering* of events

- Keep:
  - Local assumptions

    In accordance with local observations

  - Logical consistency 

enter the world of the non-causal



Background image: E. Schiele, *„Schiele mit Aktmodell vor dem Spiegel,“* (Pencil on paper, 1910)

Antinomies

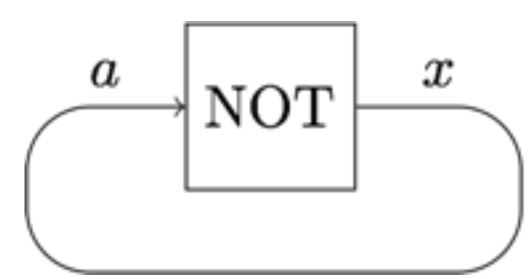
- Grandfather antinomy
  Overdetermination
  An effect suppresses its own cause

- Information antinomy
  Underdetermination
  Multiple effects confirm their own causes, yet the theory fails to predict with what probability which cause-effect pair will take place

Grandfather Antinomy

- Travel to the past and prevent the younger self from traveling to the past

- Overdetermination (contradiction):
  $x=f(a)$,
  $a=g(x)$,
  <u>no</u> pair a,x satisfies both equations



$$x = \neg a$$

$$a = x$$

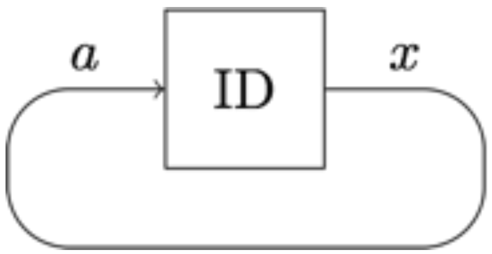
Information Antinomy

Also known as Bootstrap Antinomy

- One morning you find a book on your table, publish it, win the Fields Medal, then you travel back in time to place the book on the table.

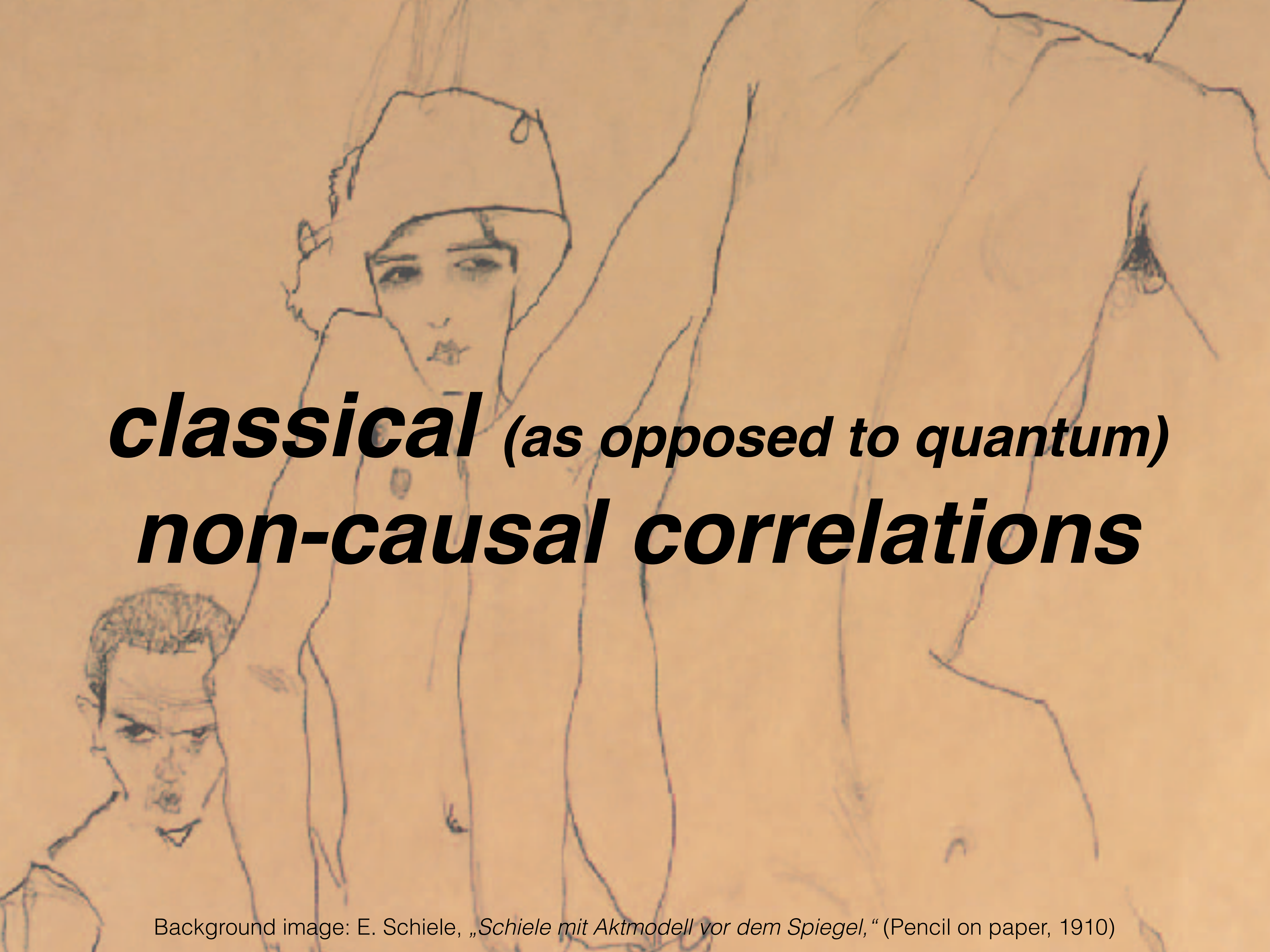
  This is *creationism*.

- Underdetermination:
  $x=f'(a)$
  $a=g'(x)$
  <u>multiple</u> pairs *a,x* satisfy both equations



$x = a$
$a = x$

D. Deutsch, *Physical Review D* **44**, 3197 (1991)



classical *(as opposed to quantum)*
non-causal correlations

Background image: E. Schiele, *„Schiele mit Aktmodell vor dem Spiegel,“* (Pencil on paper, 1910)

Process-Matrix Framework

ARTICLE

Received 29 May 2012 | Accepted 17 Aug 2012 | Published 2 Oct 2012

DOI: 10.1038/ncomms2076

Quantum correlations with no causal order

Ognyan Oreshkov[1,2], Fabio Costa[1] & Časlav Brukner[1,3]

- Drop assumption: *definite partial ordering* of events

  - Local assumptions only

    In accordance with local observations

  - Logical consistency

O. Oreshkov, F. Costa, Č. Brukner, *Nature Communications* **3**, 1092 (2012)

Classical Non-Causal Correlations

Assumptions

(1) Parties interact with random variables (not quantum)
Each party interacts once
A party is described by a stochastic operation

(2) Parties are isolated
Multiple parties: set of stochastic operations

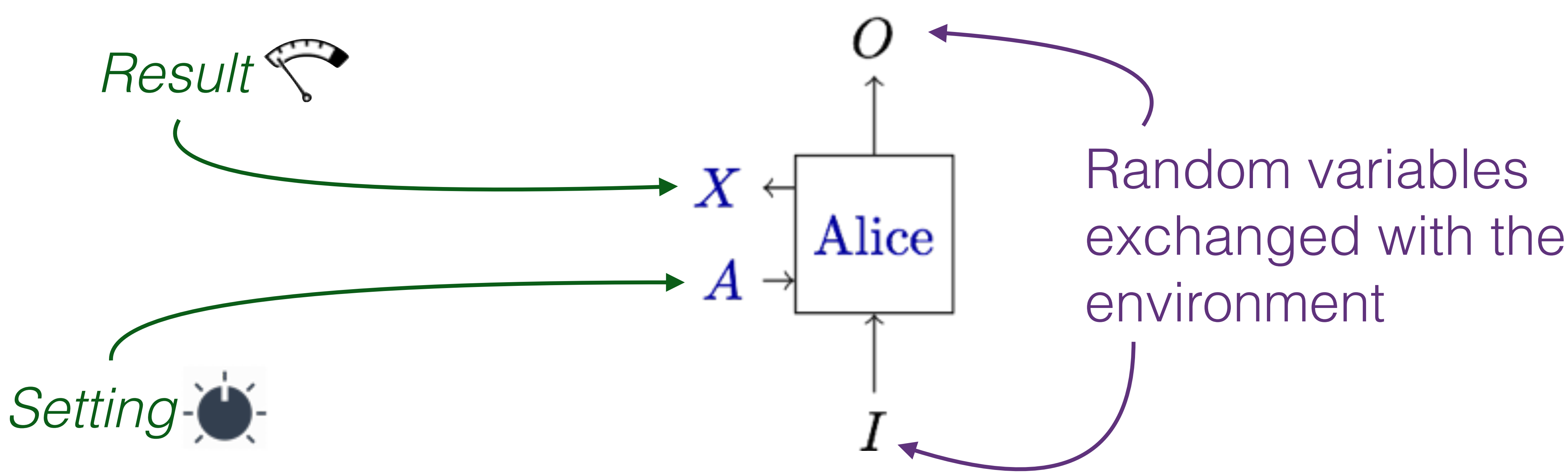
(3) Logical consistency
Probabilities are linear in the choice of operation

Ä. B., S. Wolf, *New Journal of Physics* **18**, 013036 (2016)

Classical Non-Causal Correlations

Assumptions

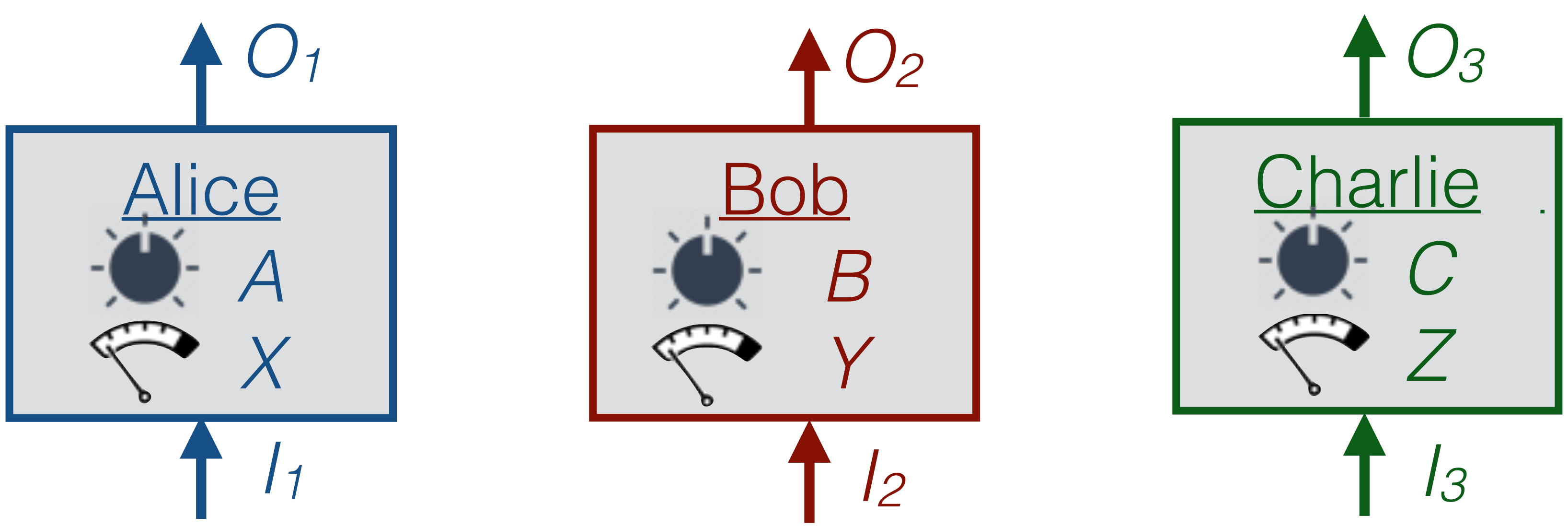
- Parties interact with random variables (as opposed to quantum systems)
  A party is described by a stochastic operation $P_{X,O|A,I} =: L$



Ä. B., S. Wolf, *New Journal of Physics* **18**, 013036 (2016)

Classical Non-Causal Correlations
Assumptions

- Parties are isolated
  Multiple parties: set of stochastic operations

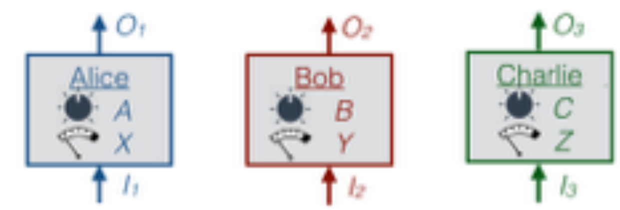


$$\{\underbrace{P_{X,O_1|A,I_1}}_{L_1}, \underbrace{P_{Y,O_2|B,I_2}}_{L_2}, \underbrace{P_{Z,O_3|C,I_3}}_{L_3}, \dots\}$$

Ä. B., S. Wolf, *New Journal of Physics* **18**, 013036 (2016)

Classical Non-Causal Correlations

Assumptions



- Logical consistency
  Probabilities are linear in the choice of local operations

$$\forall L_1, L_2, L_3, \cdots : f(L_1, L_2, L_3, \ldots) \text{ is a probability distribution}$$

Local operations (stochastic)

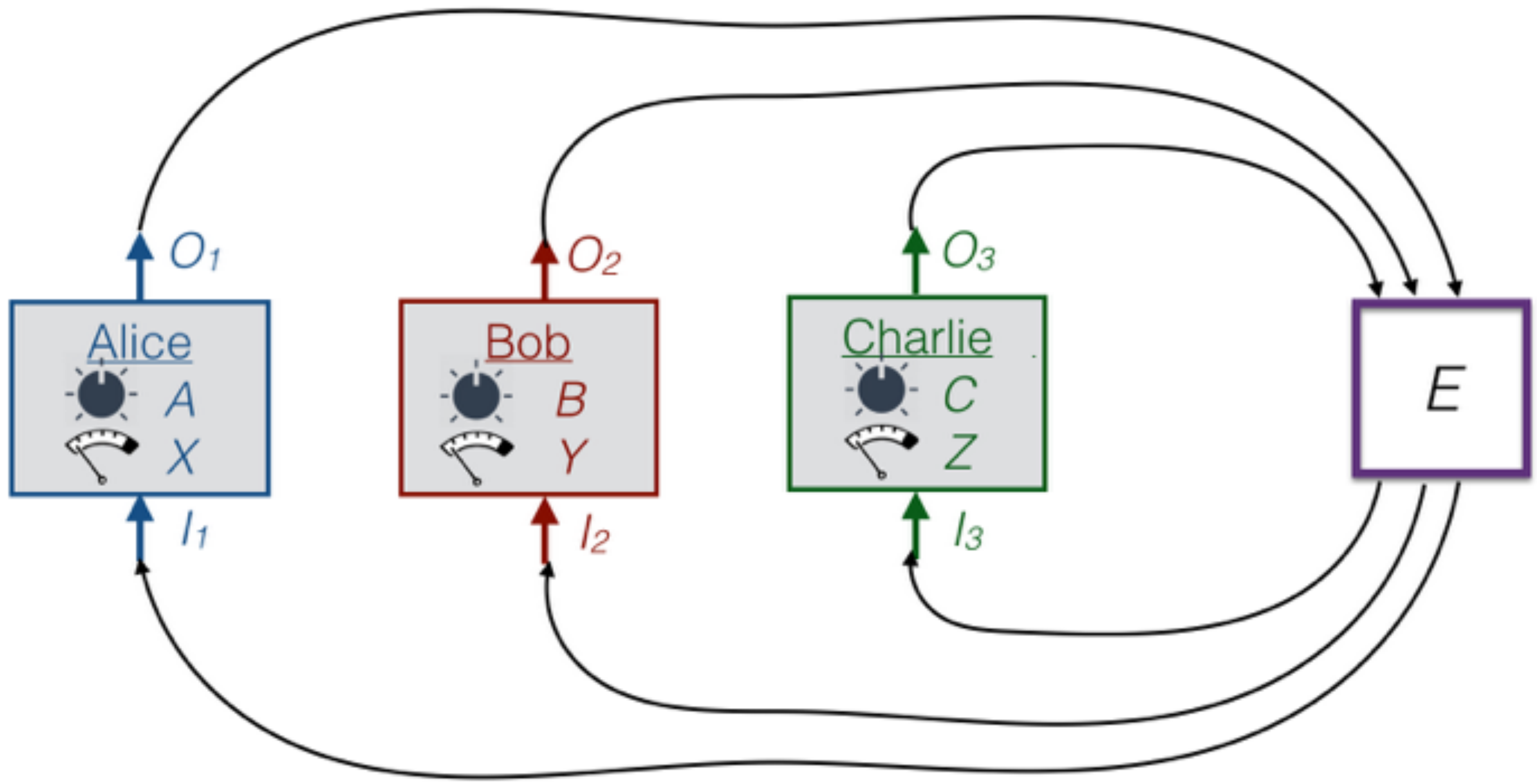
$$P_{X,Y,Z,\cdots|A,B,C,\ldots} = f(L_1, L_2, L_3, \ldots)$$

Linear in choice of local operations

Ä. B., S. Wolf, *New Journal of Physics* **18**, 013036 (2016)

Classical Non-Causal Correlations

Theorem



$$P_{X,Y,Z|A,B,C} = \sum_{\substack{i_1,i_2,i_3\\o_1,o_2,o_3}} P_{X,O_1|A,I_1} P_{Y,O_2|B,I_2} P_{Z,O_3|B,I_3} \underbrace{P_{I_1,I_2,I_3|O_1,O_2,O_3}}_{E}$$

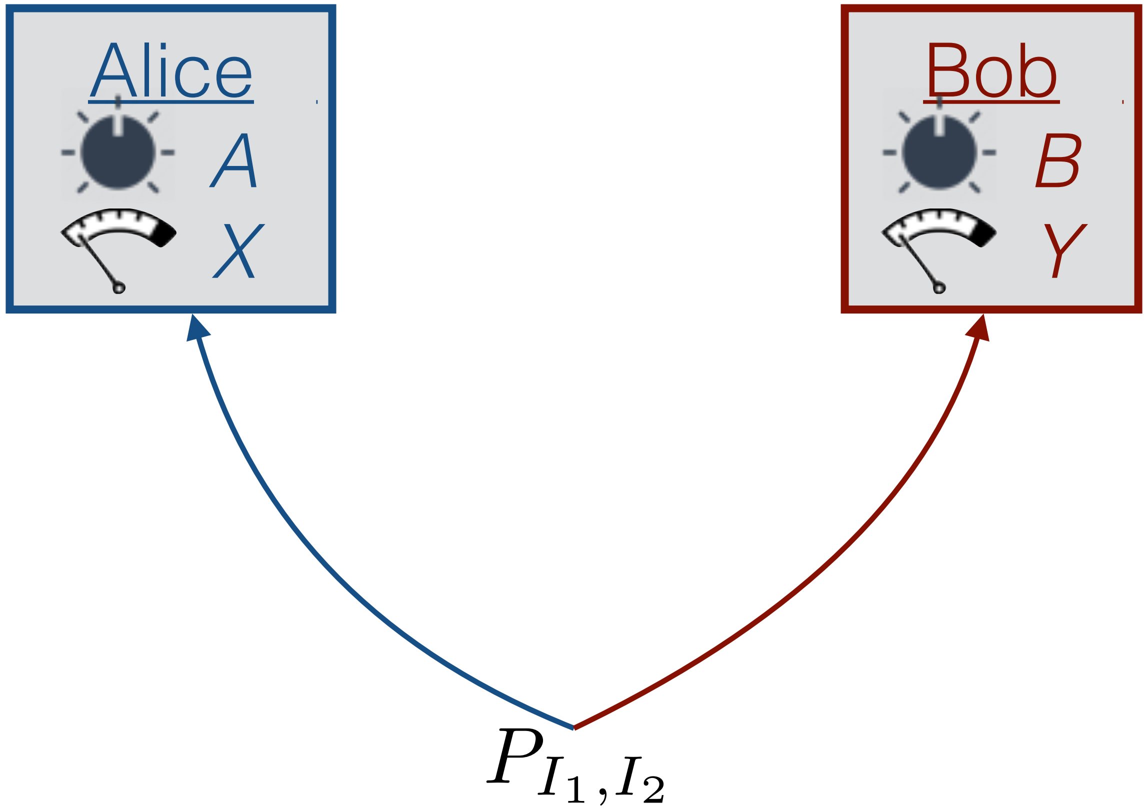
$$E = P_{I_1,I_2,I_3,\dots|O_1,O_2,O_3,\dots}$$

with some restrictions

Ä. B., S. Wolf, *New Journal of Physics* **18**, 013036 (2016)

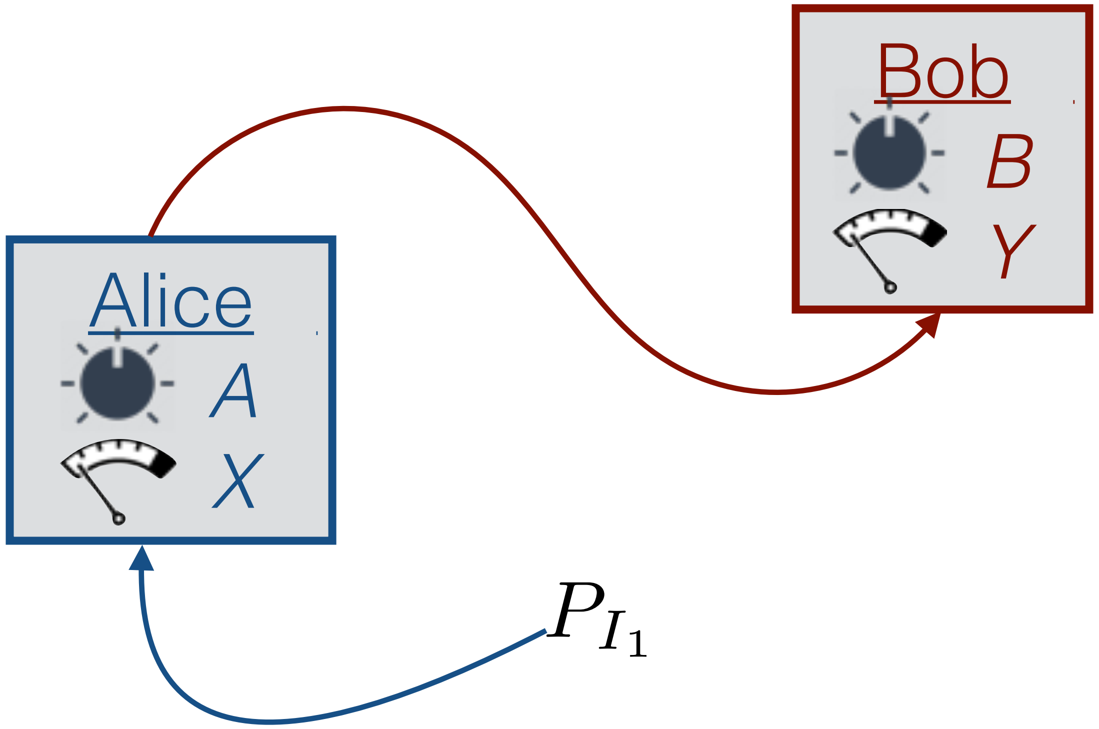
Examples

- <u>Shared State:</u>

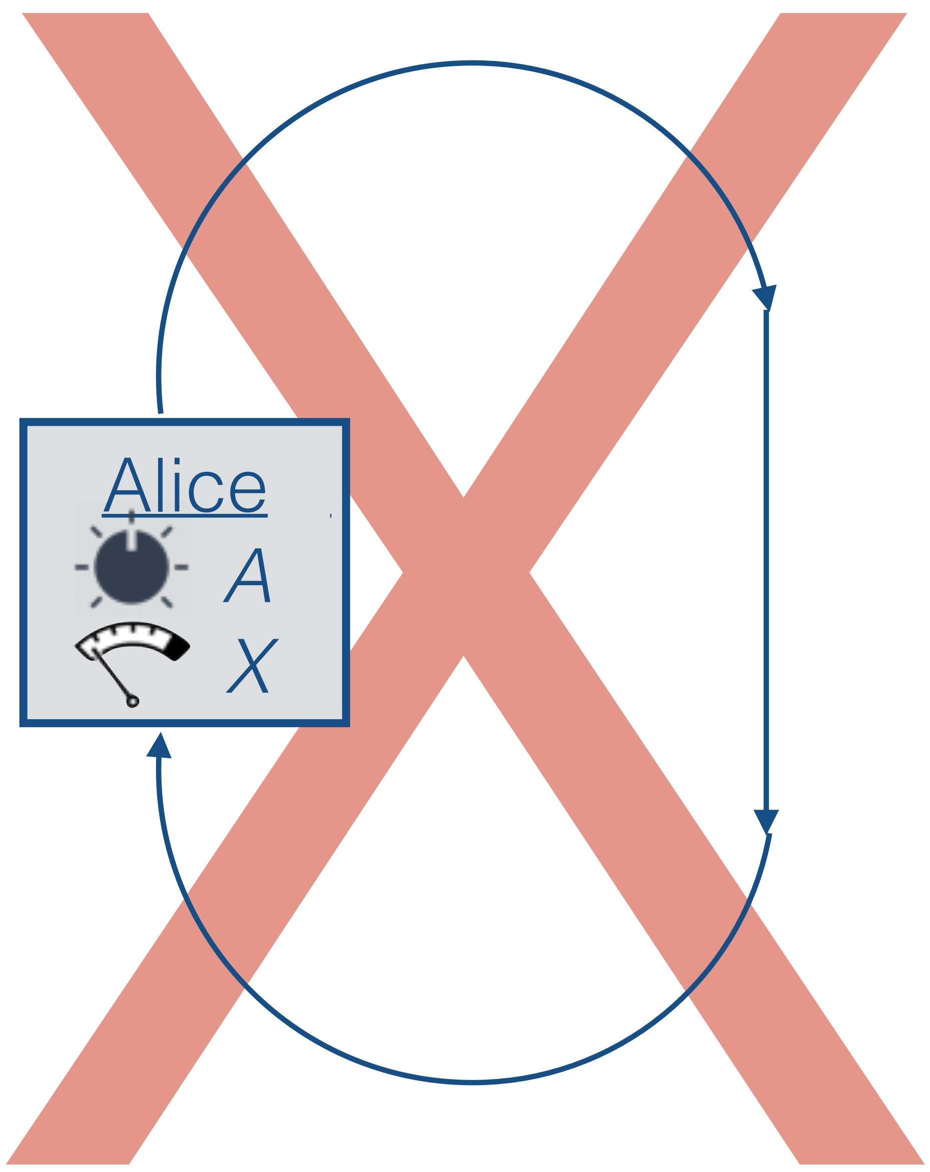


Examples

- <u>Channel:</u>

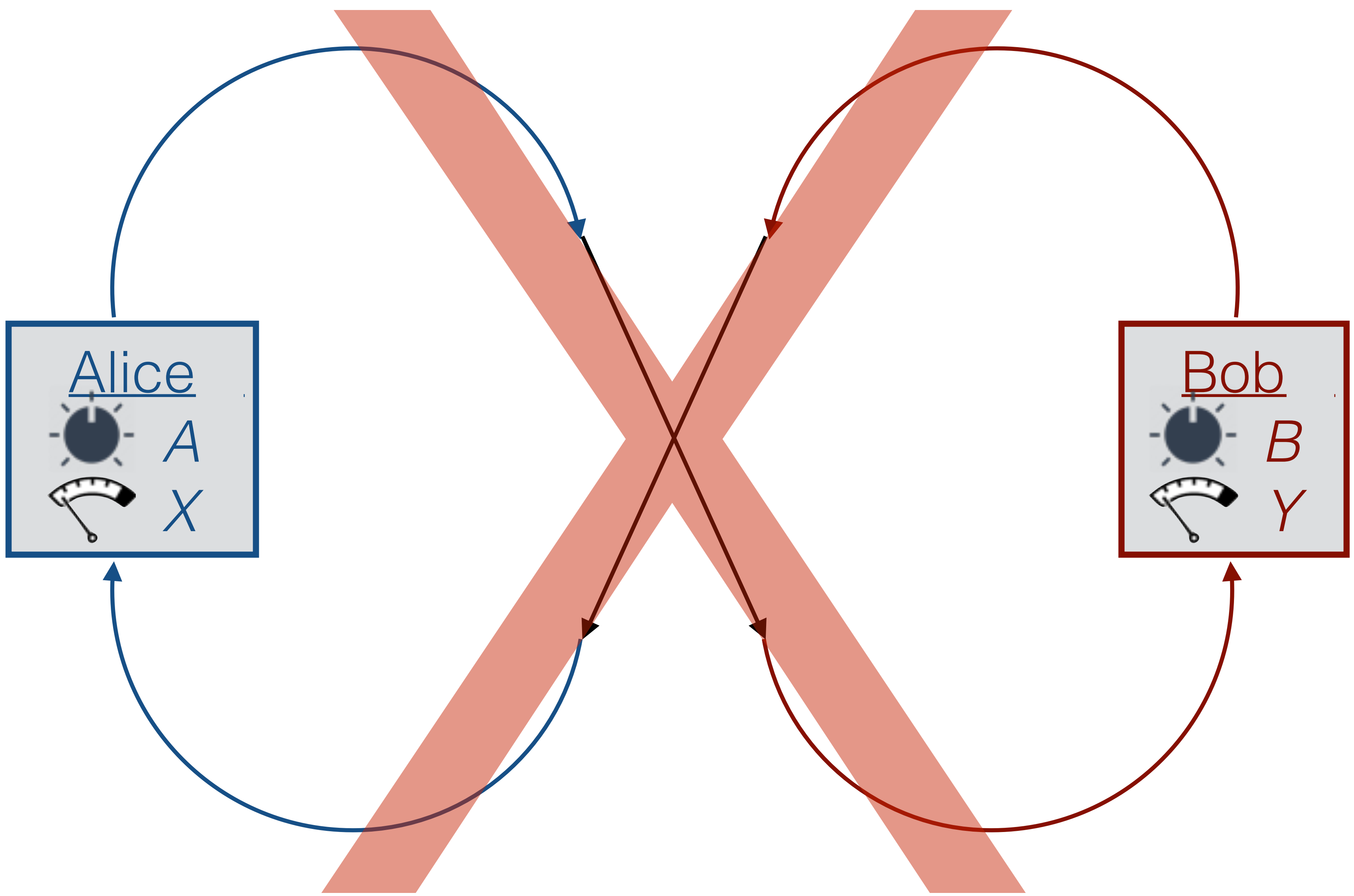


Classical Non-Causal Correlations



O. Oreshkov, F. Costa, Č. Brukner, *Nature Communications* **3**, 1092 (2012)

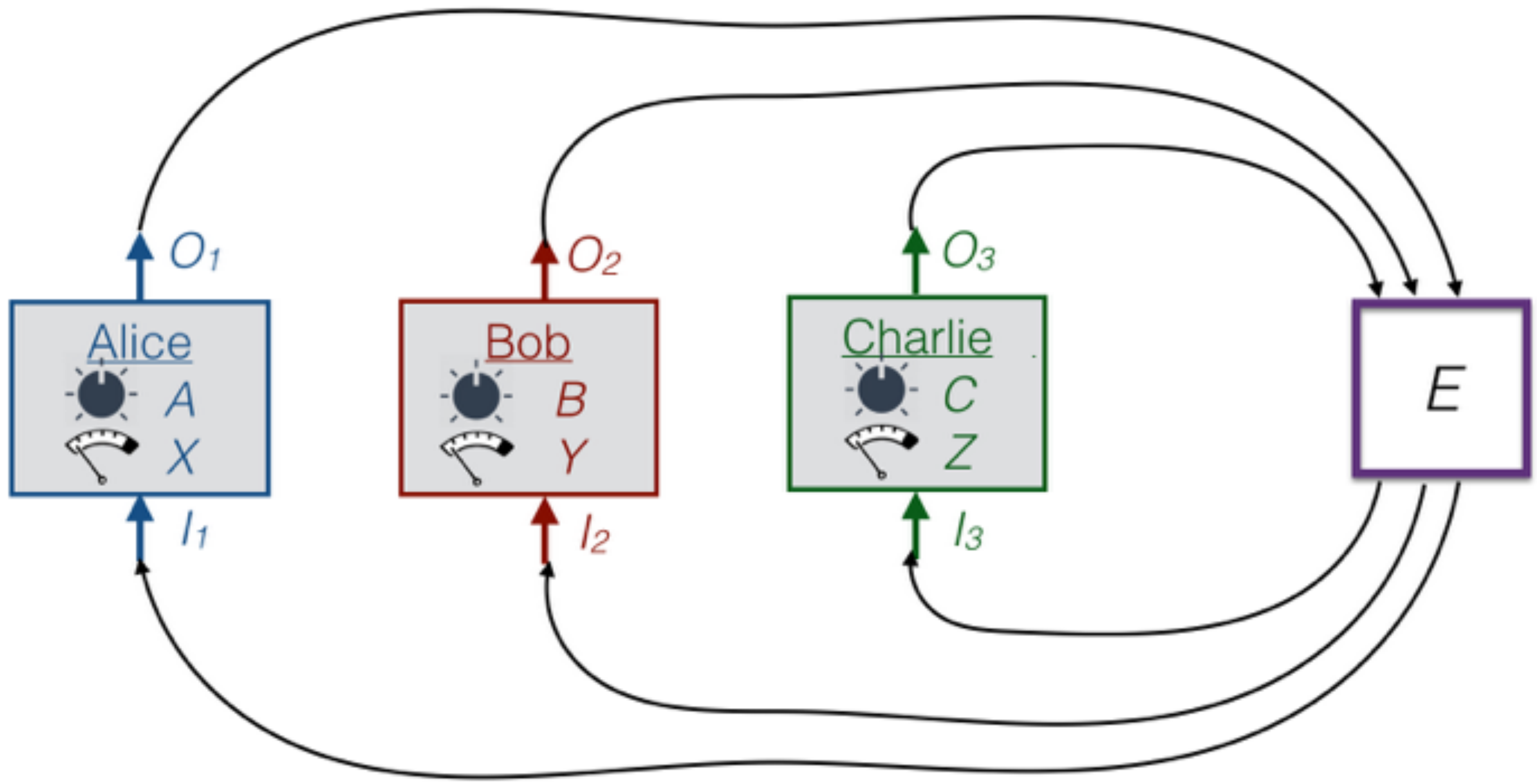
Classical Non-Causal Correlations



O. Oreshkov, F. Costa, Č. Brukner, *Nature Communications* **3**, 1092 (2012)

Classical Non-Causal Correlations

What else is possible?



$$P_{X,Y,Z|A,B,C} = \sum_{\substack{i_1,i_2,i_3\\o_1,o_2,o_3}} P_{X,O_1|A,I_1} P_{Y,O_2|B,I_2} P_{Z,O_3|B,I_3} \underbrace{P_{I_1,I_2,I_3|O_1,O_2,O_3}}_{E}$$

$$E = P_{I_1,I_2,I_3,\cdots|O_1,O_2,O_3,\ldots}$$

with some restrictions

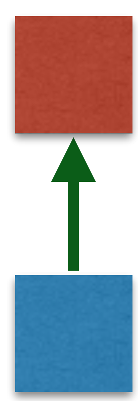
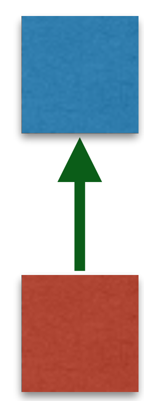
violation of causal inequalities

Background image: E. Schiele, *„Schiele mit Aktmodell vor dem Spiegel,“* (Pencil on paper, 1910)

Causal Correlations

- Correlations among parties $P_{X,Y|A,B}$

- **Definition (Causal Correlations):**
  Correlations obtainable from a predefined partial ordering of the parties

- For two parties:  or  or 

O. Oreshkov, F. Costa, Č. Brukner, *Nature Communications* **3**, 1092 (2012)

Causal Correlations

- Correlations among parties $P_{X,Y|A,B}$

- **Definition (Causal Correlations):**
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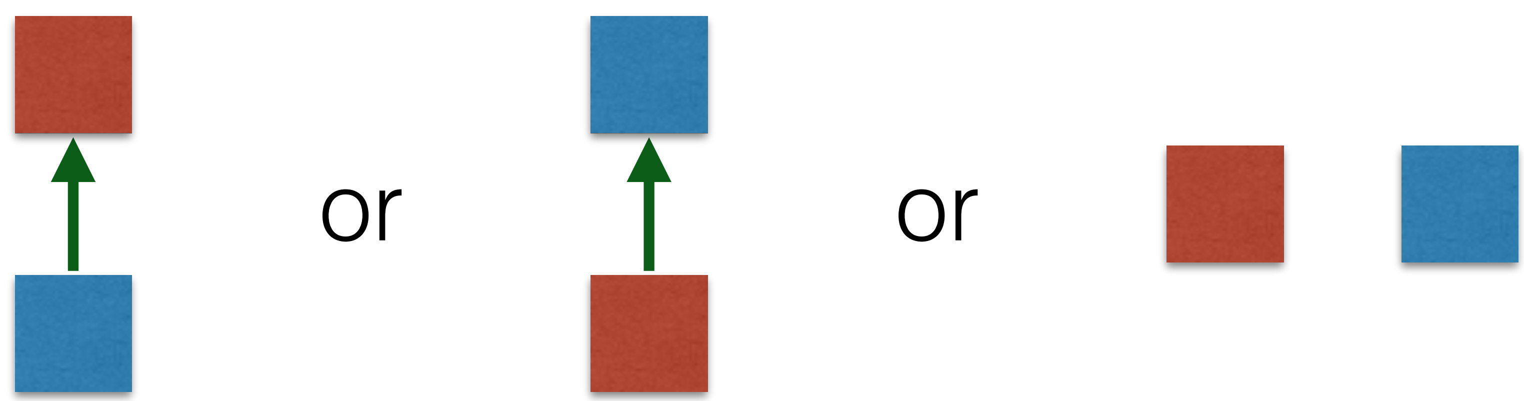
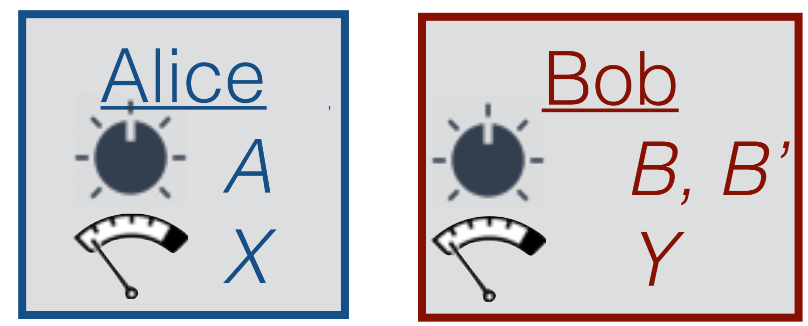


$$P_{X,Y|A,B} = pP_{X|A}P_{Y|A,B,X} + (1-p)P_{X|A,B,Y}P_{Y|B}$$

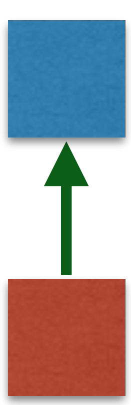
O. Oreshkov, F. Costa, Č. Brukner, *Nature Communications* **3**, 1092 (2012)

Causal Inequalities

- Inequalities satisfied by all causal correlations
- Example:

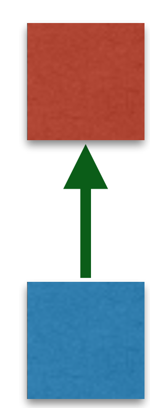


$$\frac{1}{2}\Pr(X = B \mid B' = 0) + \frac{1}{2}\Pr(Y = A \mid B' = 1) \leq \frac{3}{4}$$



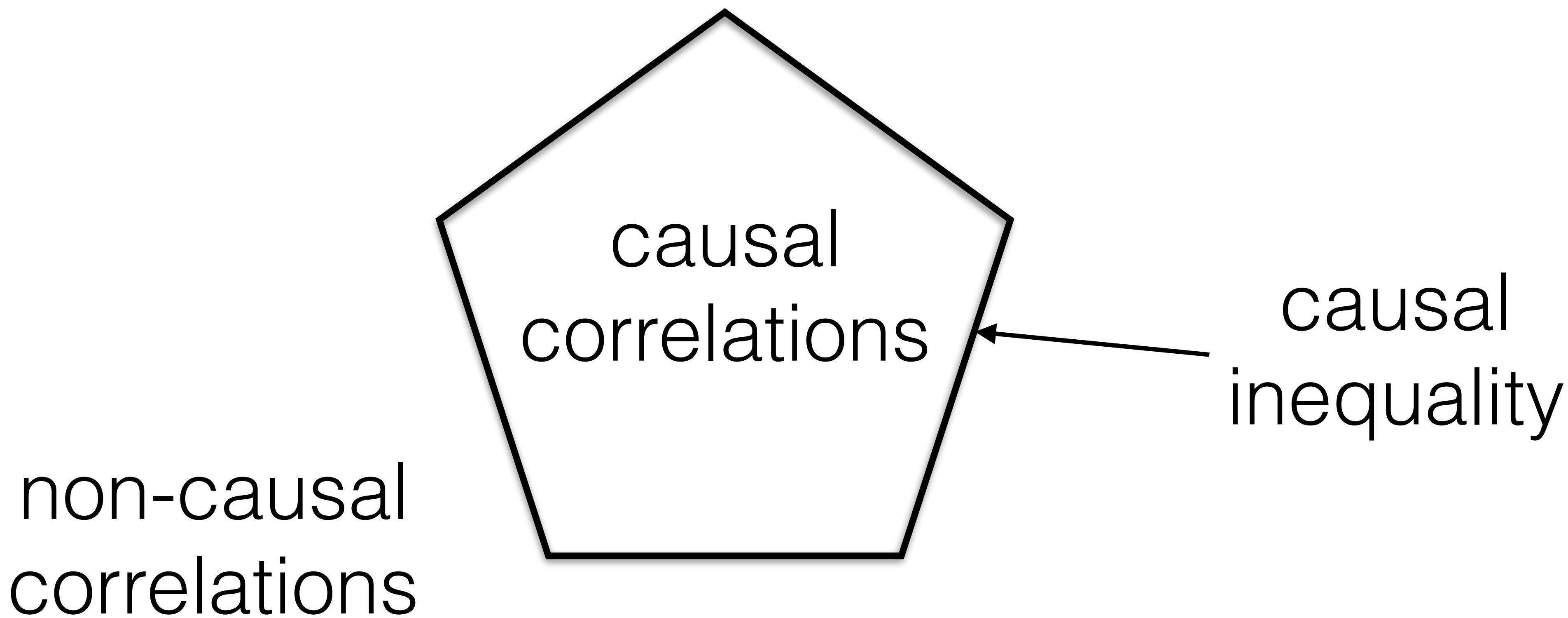
Bob before Alice

Alice before Bob

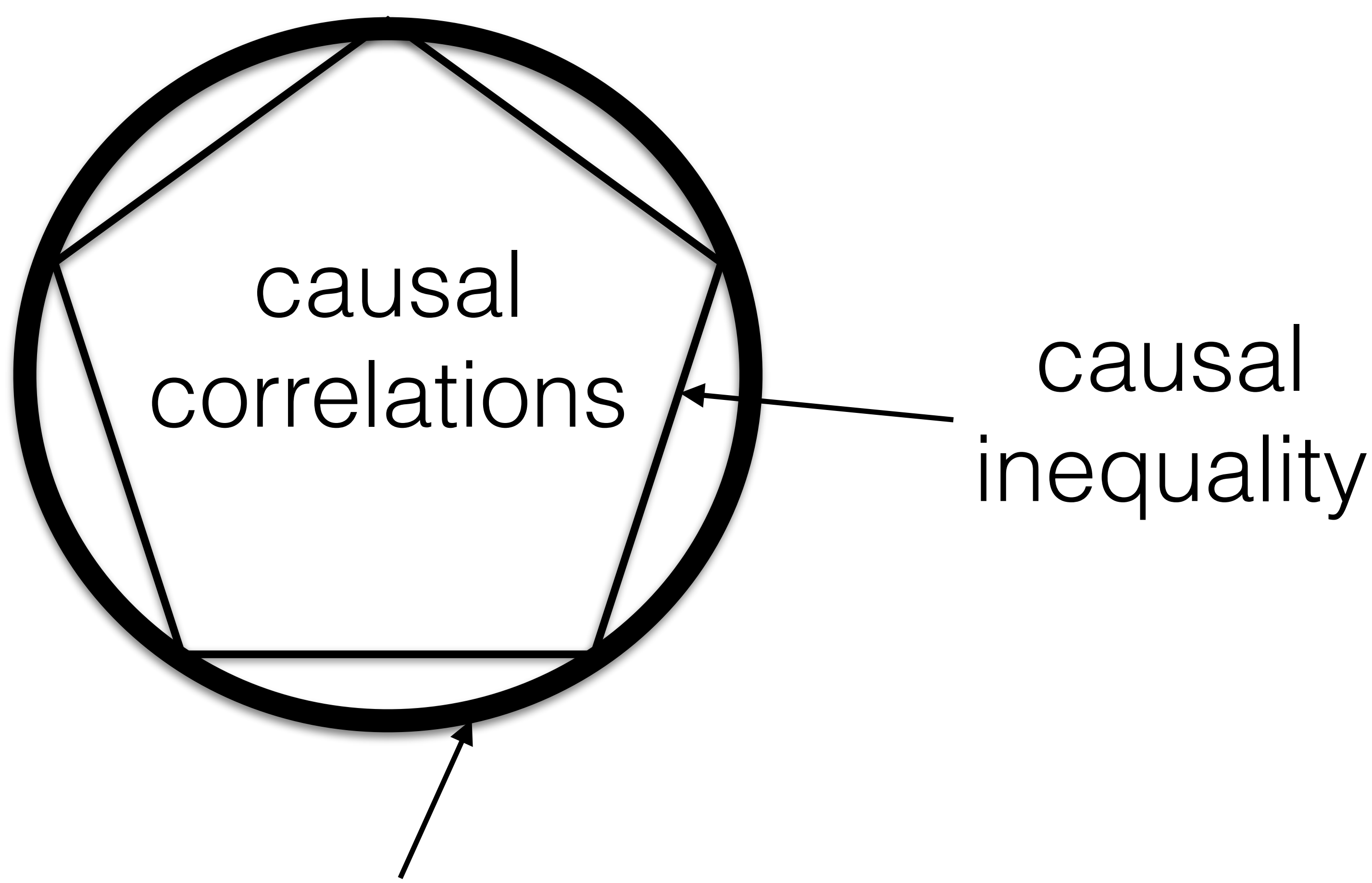


O. Oreshkov, F. Costa, Č. Brukner, *Nature Communications* **3**, 1092 (2012)

Causal Inequalities



Causal Inequalities



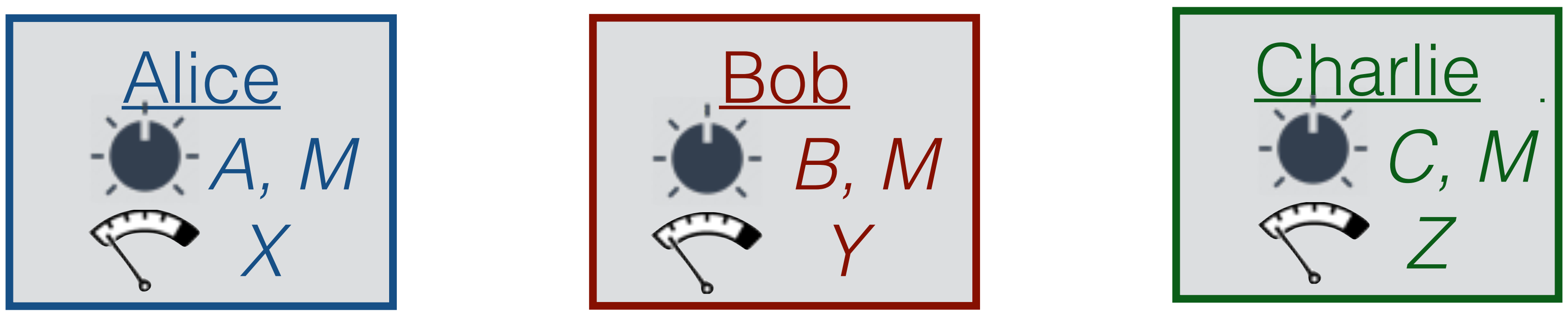
classical logically consistent correlations

=

causal correlations?

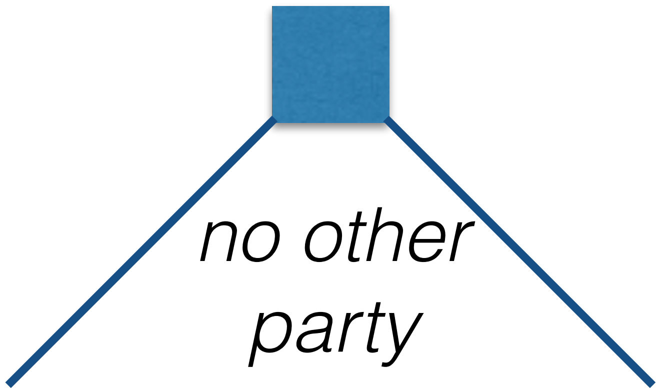
Classical Non-Causal Correlations

Non-Causal Environment



$$\frac{1}{3}(\Pr(X = B \oplus C \mid M = 1) + \Pr(Y = A \oplus C \mid M = 2) + \Pr(Z = A \oplus B \mid M = 3)) \leq \frac{5}{6}$$

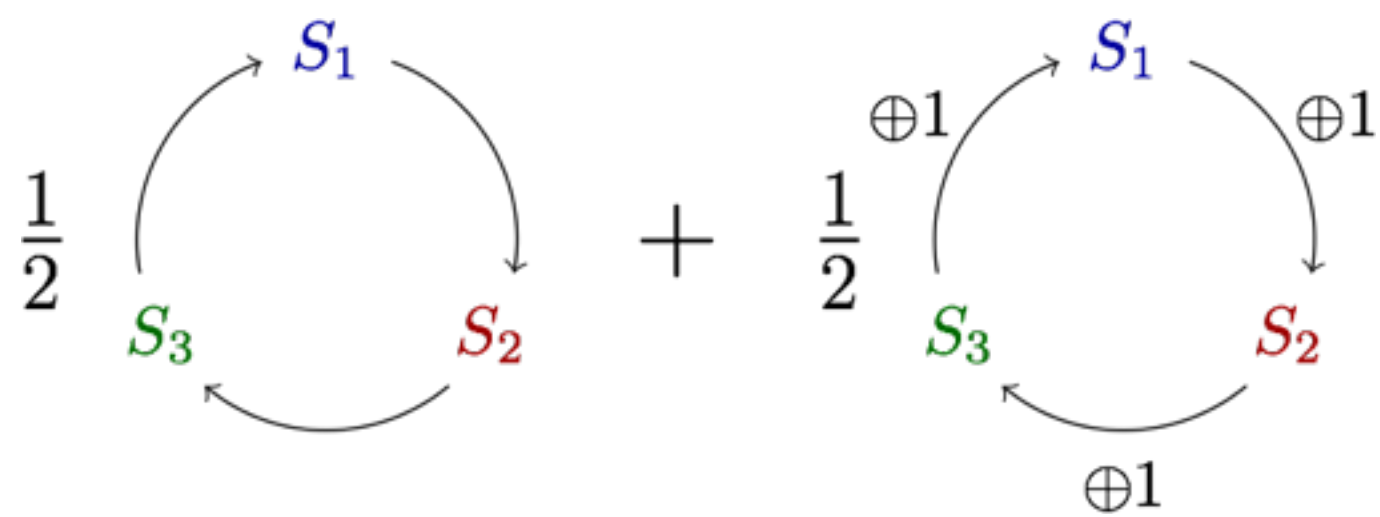
Causal:



Ä. B., A. Feix, S. Wolf, *Physical Review A* **90**, 042106 (2014)

Classical Non-Causal Correlations

Non-Causal Environment



$$\frac{1}{3}\left(\Pr(X = B \oplus C \mid M = 1) + \Pr(Y = A \oplus C \mid M = 2) + \Pr(Z = A \oplus B \mid M = 3)\right) \leq \frac{5}{6}$$

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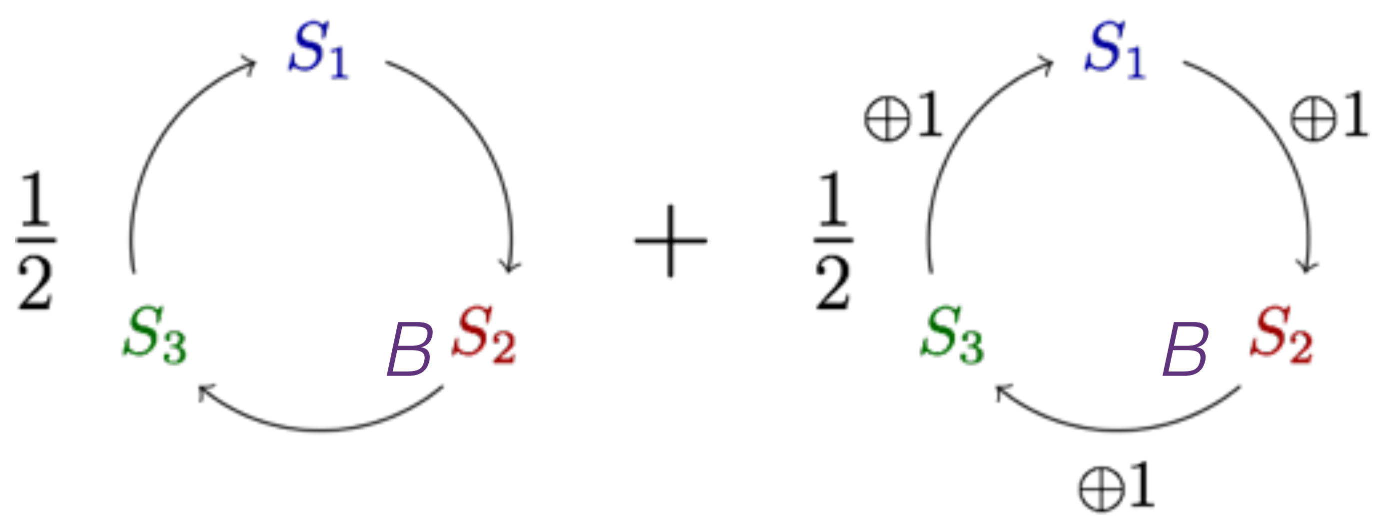
Classical Non-Causal Correlations

Non-Causal Environment



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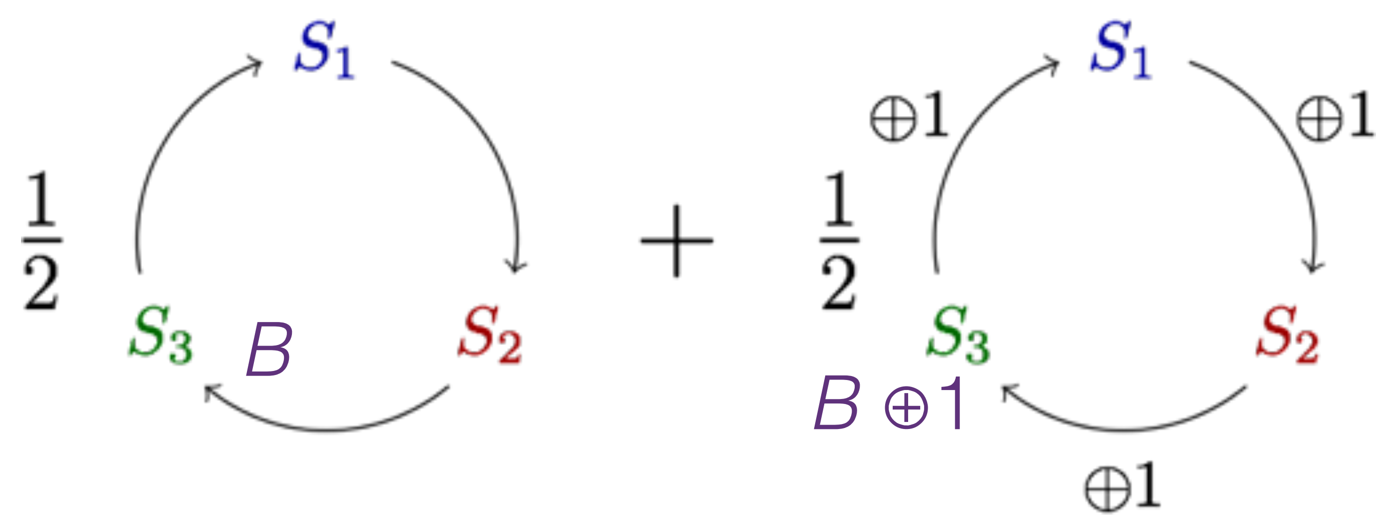
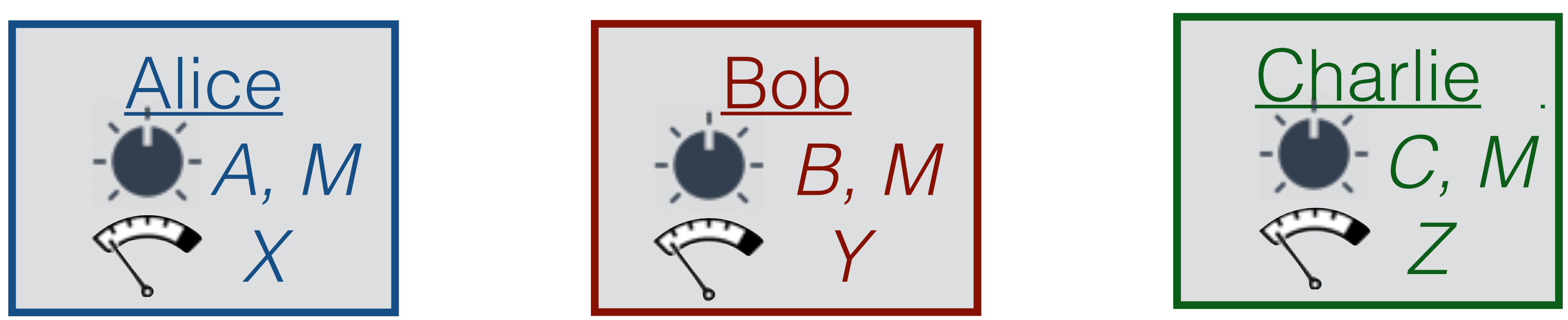
Classical Non-Causal Correlations

Non-Causal Environment





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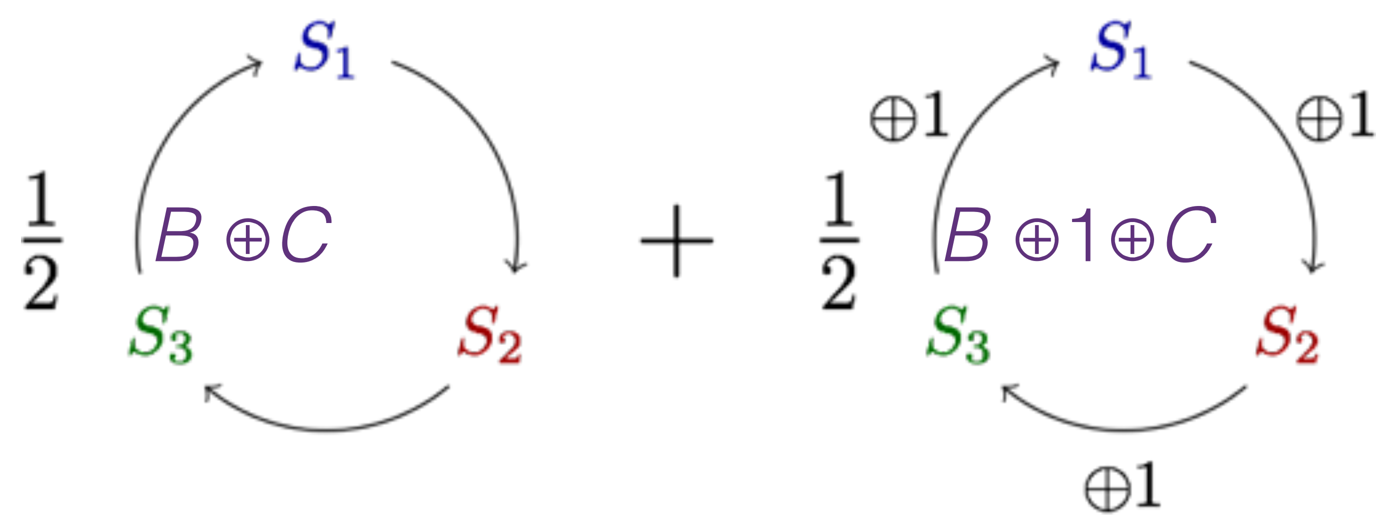
Classical Non-Causal Correlations

Non-Causal Environment



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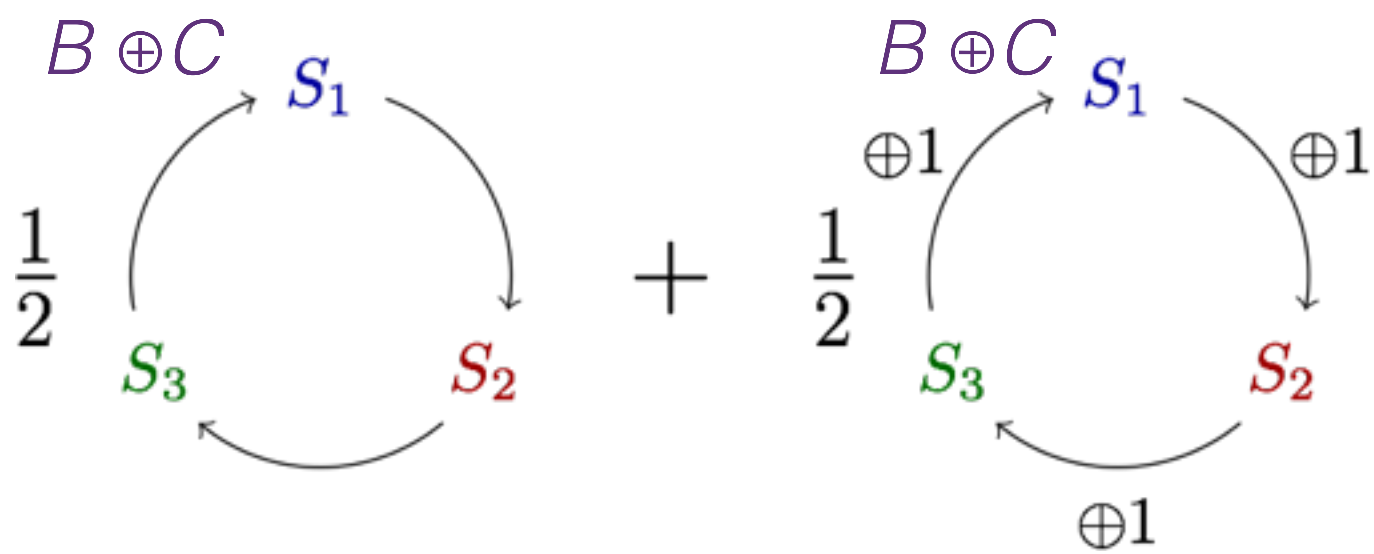
Ä. B., A. Feix, S. Wolf, *Physical Review A* **90**, 042106 (2014)

Classical Non-Causal Correlations

Non-Causal Environment



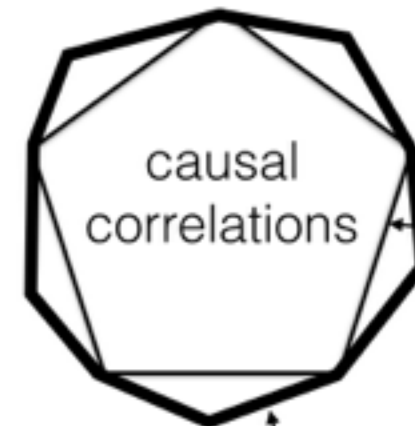
$$\frac{1}{3}(\Pr(X = B \oplus C \mid M = 1) + \Pr(Y = A \oplus C \mid M = 2) + \Pr(Z = A \oplus B \mid M = 3)) \leq \frac{5}{6}$$

Ä. B., A. Feix, S. Wolf, *Physical Review A* **90**, 042106 (2014)

Classical Non-Causal Correlations

Characterizing the environment

- Characterization with polytopes



- Characterization with fixed-point theorems
  - No fixed point: Grandfather antinomy
  - Multiple fixed points: Information antinomy

  For every choice of operation:

  => deterministic case: *unique* fixed point

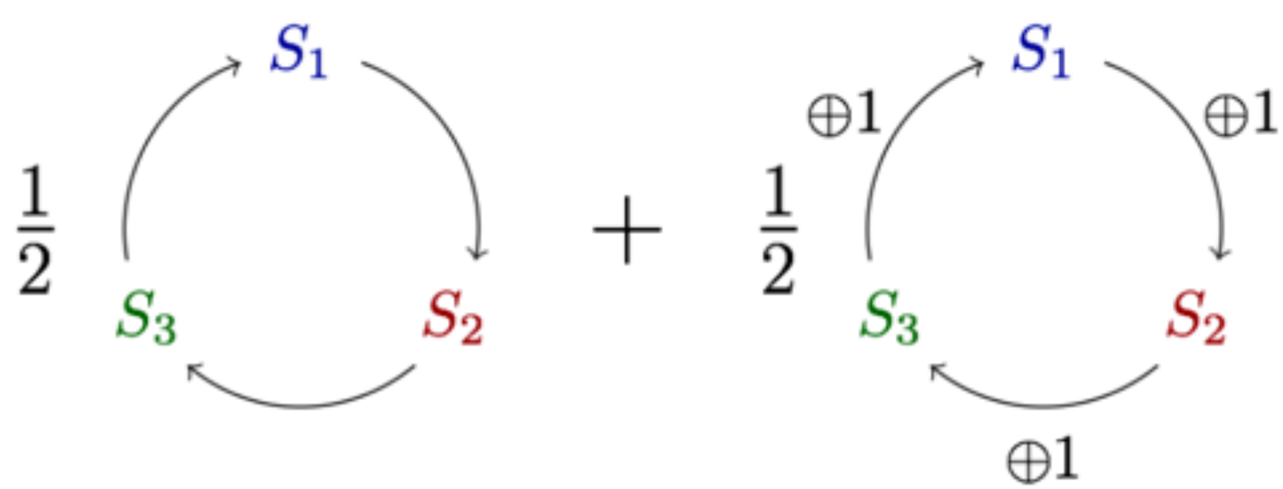
  => probabilistic case: *average* number of fixed points is 1

Ä. B., S. Wolf, *New Journal of Physics* **18**, 013036 (2016); Ä. B., S. Wolf, *New Journal of Physics* **18**, 035014 (2016)

Classical Non-Causal Correlations

Characterizing the environment

- Characterization with polytopes



Number of fixed points, where all parties use identity operation: 2 0

For every choice of operation:

=> deterministic case: *unique* fixed point

=> probabilistic case: *average* number of fixed points is 1

Ä. B., S. Wolf, *New Journal of Physics* **18**, 013036 (2016); Ä. B., S. Wolf, *New Journal of Physics* **18**, 035014 (2016)

non-causal computation

Background image: E. Schiele, *„Schiele mit Aktmodell vor dem Spiegel,“* (Pencil on paper, 1910)

Non-Causal Computation

<u>Before:</u>

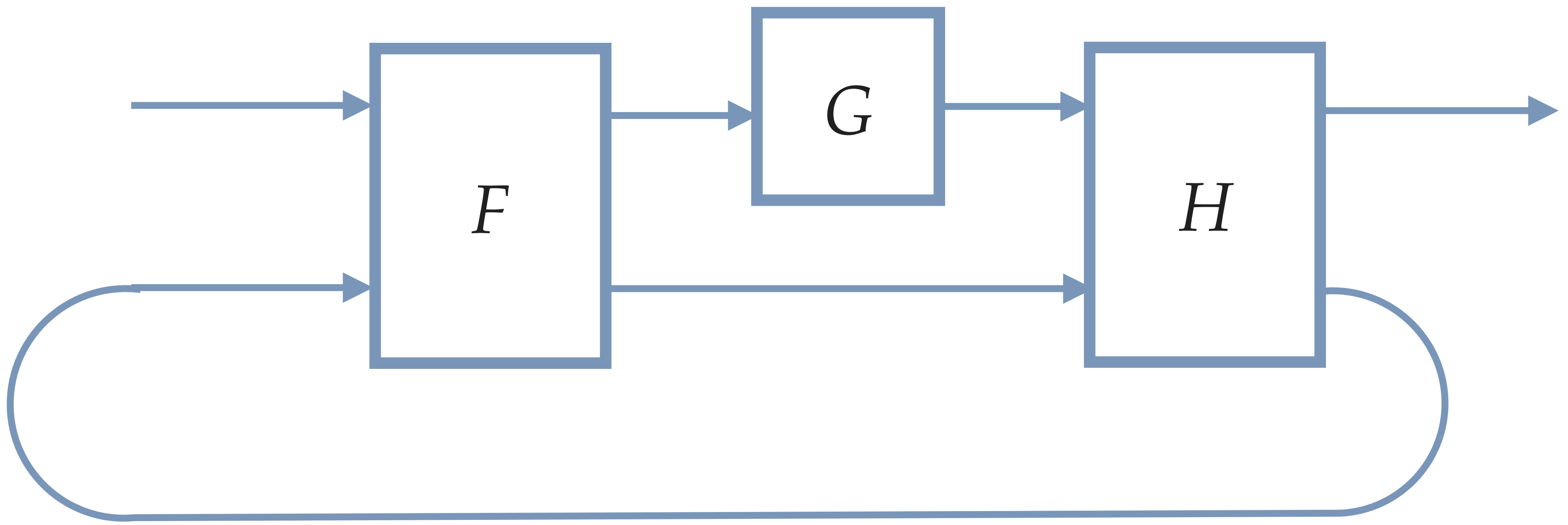
- Parties
- Order not fixed
- Logical consistency:
  $\forall L_1, L_2, L_3$ unique F.P.

<u>Model of computation:</u>

- Gates (deterministic)
- Arbitrary wiring
- Logical consistency:
  for every input: loops in circuit have unique F.P.

Non-Causal Computation

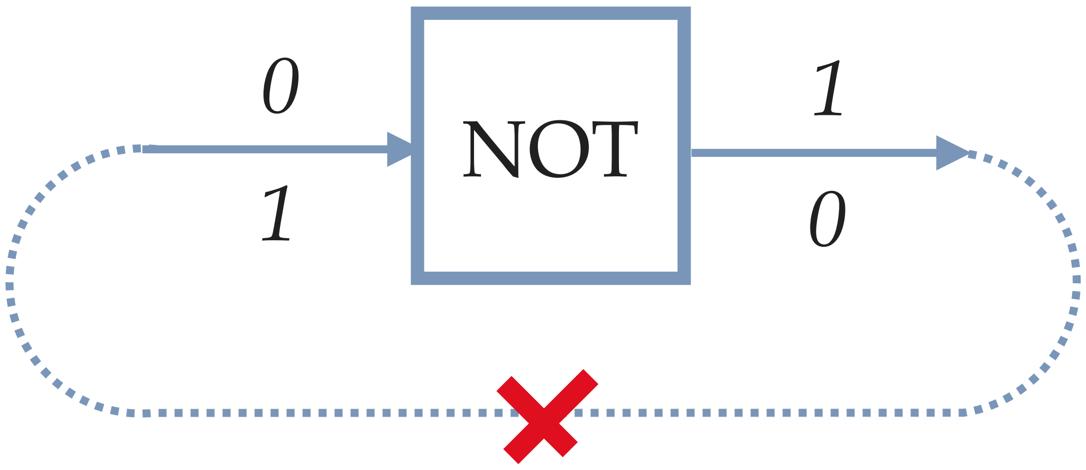
- Arbitrary wiring of gates



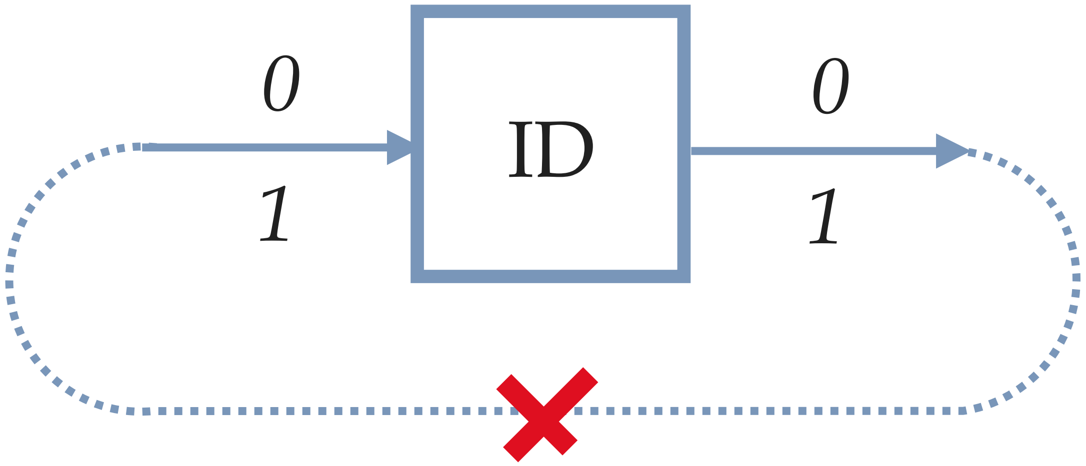
- Logical consistency:
  Unique fixed point on looping wires

Non-Causal Computation

- Not all wirings are logically consistent



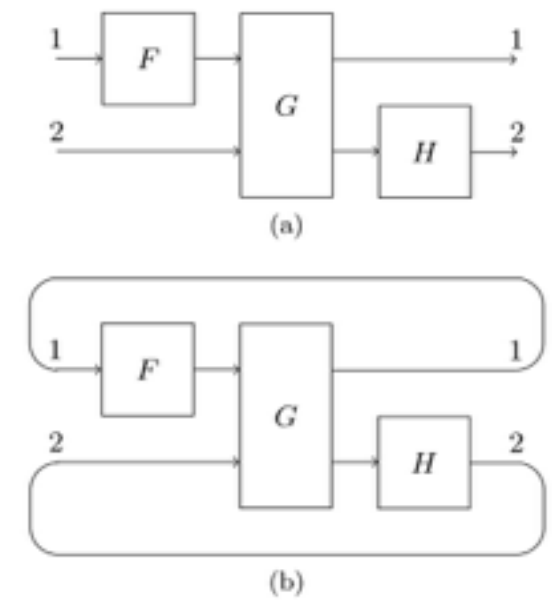
#fixed-points: 0



#fixed-points: 2

Non-Causal Computation

- Language: $L \subseteq \{0,1\}^*$
- Instance: $x \in \{0,1\}^*$
  Question: $x \in L$?



- <u>Definition (NCCAlgo):</u>
  A deterministic NCCAlgo *A* is a polytime algorithm that takes as input *x* and outputs a Boolean circuit c_x over AND, OR, NOT such that:

$$\forall x \in \{0,1\}^*, \exists! y : c_x(y) = y$$

  If *y=1z*: *A* *<u>accepts</u>* *x,* otherwise *A* *<u>rejects</u>* *x.*
  A *<u>decides</u>* *L* if it accepts every *x* in *L* and rejects every other *x*

Ä.B., S. Wolf, *Entropy* **19**, 326 (2017); Ä. B., S. Wolf, *Proc. Royal Soc. A* **474**, 20170698 (2018)

Non-Causal Computation

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- Definition (P_{NCC}):
  The class P_{NCC} contains all languages decidable by some NCCAlgo *A.*

Ä.B., S. Wolf, *Entropy* **19**, 326 (2017); Ä. B., S. Wolf, *Proc. Royal Soc. A* **474**, 20170698 (2018)

Non-Causal Computation

- Definition (P_{NCC}):
  The class P_{NCC} contains all languages decidable by some NCCAlgo *A.*

- Definition (UP ∩ coUP):
  The class UP ∩ coUP contains all languages *L* for which there exist two polytime verifiers
  V_{yes}: {0,1}* x {0,1}* —> {0,1}
  V_{no}: {0,1}* x {0,1}* —> {0,1}
  such that:

$$x \in L \implies \exists! y : V_{\text{yes}}(x,y) = 1 \quad \wedge \quad \forall y : V_{\text{no}}(x,y) = 0$$
$$x \notin L \implies \forall y : V_{\text{yes}}(x,y) = 0 \quad \wedge \quad \exists! y : V_{\text{no}}(x,y) = 1$$

L.G. Valiant, *Inf. Proc. Lett.* **5**, 20 (1976);
Ä.B., S. Wolf, *Entropy* **19**, 326 (2017); Ä. B., S. Wolf, *Proc. Royal Soc. A* **474**, 20170698 (2018)

Non-Causal Computation

- Definition (P_{NCC}):
  The class P_{NCC} contains all languages decidable by some NCCAlgo *A.*

- Definition (UP ∩ coUP):
  The class UP ∩ coUP contains all languages *L* for which there exist two polytime verifiers
  V_{yes}: $\{0,1\}^* \times \{0,1\}^* \longrightarrow \{0,1\}$
  V_{no}: $\{0,1\}^* \times \{0,1\}^* \longrightarrow \{0,1\}$
  such that:

$$\begin{array}{lll} x \in L \implies & \exists! y : V_{\mathrm{yes}}(x,y) = 1 & \wedge \quad \forall y : V_{\mathrm{no}}(x,y) = 0 \\ x \notin L \implies & \forall y : V_{\mathrm{yes}}(x,y) = 0 & \wedge \quad \exists! y : V_{\mathrm{no}}(x,y) = 1 \\ & \text{UP} & \qquad \text{coUP} \end{array}$$

L.G. Valiant, *Inf. Proc. Lett.* **5**, 20 (1976);
Ä.B., S. Wolf, *Entropy* **19**, 326 (2017); Ä. B., S. Wolf, *Proc. Royal Soc. A* **474**, 20170698 (2018)

Non-Causal Computation

- <u>Theorem:</u> P_{NCC} = UP ∩ coUP

- Proof sketch:
  ⊆: We can translate a Circuit c_X into the verifiers:

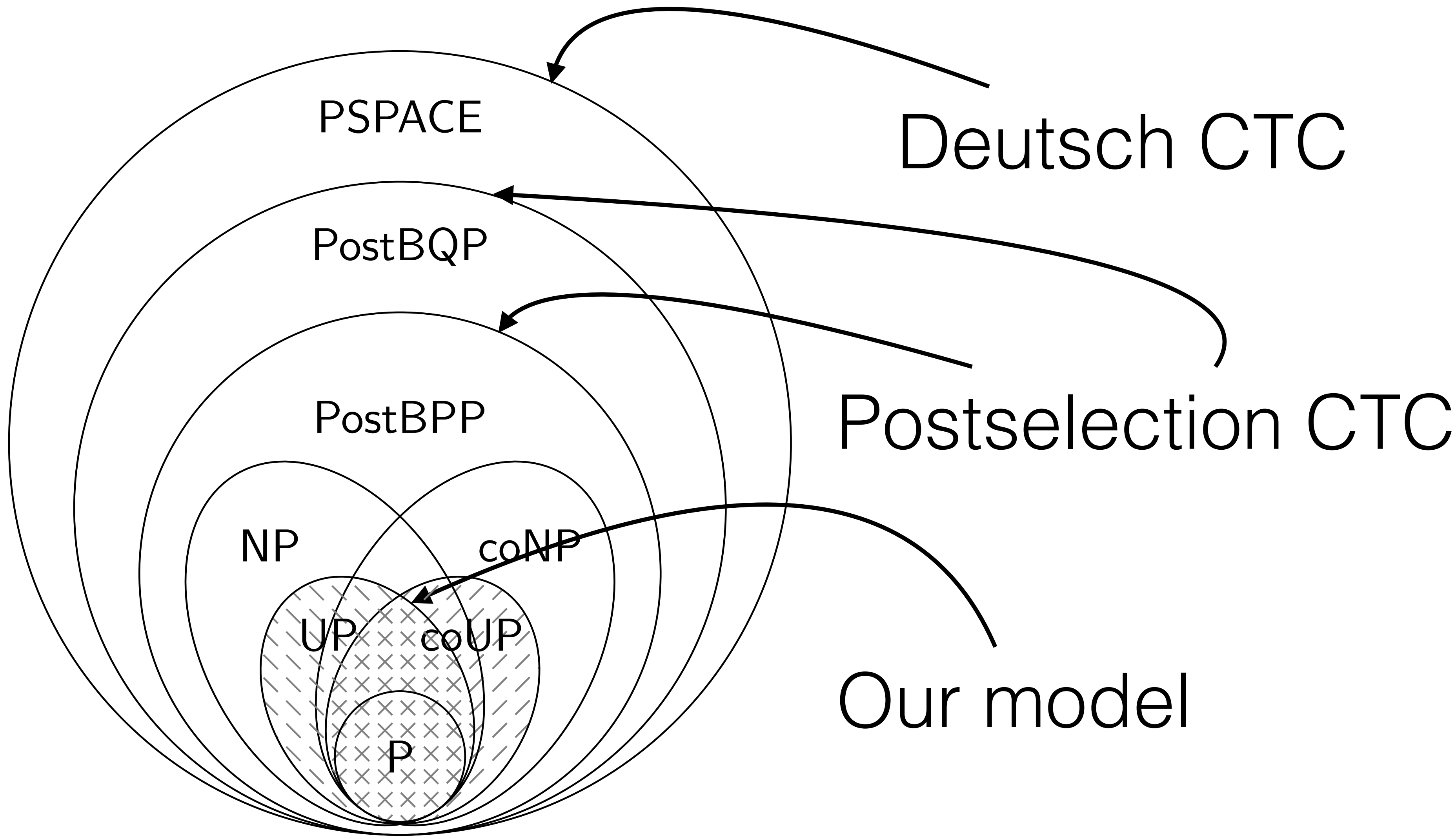
$$V_{\text{yes}}:(x,z) \mapsto c_x(z) = z \wedge z = 1w\,,$$
$$V_{\text{no}}:(x,z) \mapsto c_x(z) = z \wedge z = 0w\,.$$

  ⊇: We can construct c_X from the verifiers:

$$c_x : \{0,1\} \times \{0,1\}^{q(|x|)} \to \{0,1\} \times \{0,1\}^{q(|x|)}\,,$$
$$: (b,w) \mapsto \begin{cases} (0,w) & V_{\text{no}}(x,w) = 1\,, \\ (1,w) & V_{\text{yes}}(x,w) = 1\,, \\ (b \oplus 1, w) & \text{otherwise,} \end{cases}$$

Ä. B., S. Wolf, *Proc. Royal Soc. A* **474**, 20170698 (2018)

Non-Causal Computation



Known problems in UP ∩ coUP: Factorization

Ä. B., S. Wolf, *Proc. Royal Soc. A* **474**, 20170698 (2018)

time travel

Background image: E. Schiele, *„Schiele mit Aktmodell vor dem Spiegel,“* (Pencil on paper, 1910)

Time Travel

Logically problematic?
- Grandfather antinomy
- Information antinomy

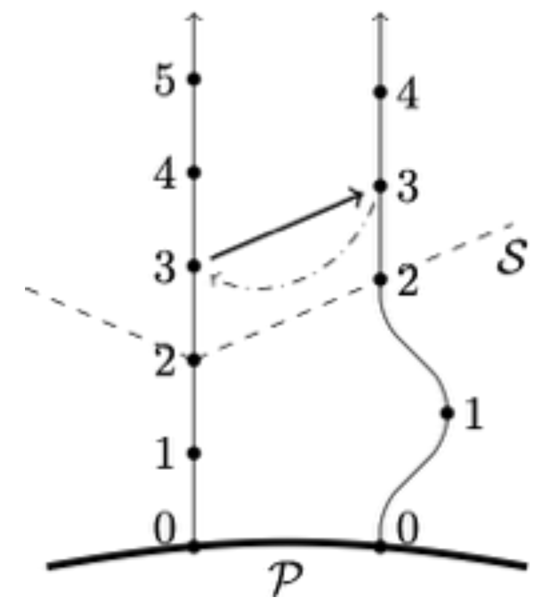
Computationally problematic?
- NP-Hardness assumption

Physically problematic?
- Reversibility of deterministic laws
- No new physics

Time Travel
(previous works)

- <u>Assumptions:</u>
  Novikov's principle of self-consistency
  (no grandfather antinomy)

  No „new physics“ at the surface *P*

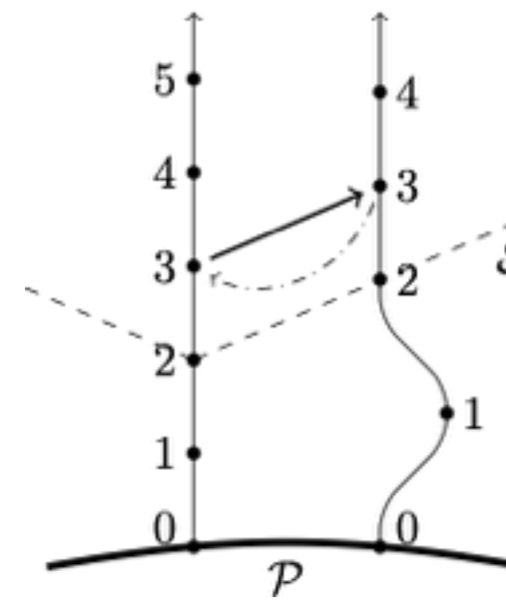


J. Friedman, M. Morris, I. Novikov, F. Echeverria, G. Klinkhammer, K. Thorne, U. Yurtsever, *Physical Review D* **42**, 1915 (1990);
F. Echeverria, G. Klinkhammer, K. Thorne, *Physical Review D* **44**, 1077 (1991);
K. Thorne, *Black Holes and Time Warps* (W. W. Norton & Company, New York, 1994)

Time Travel
(previous works)

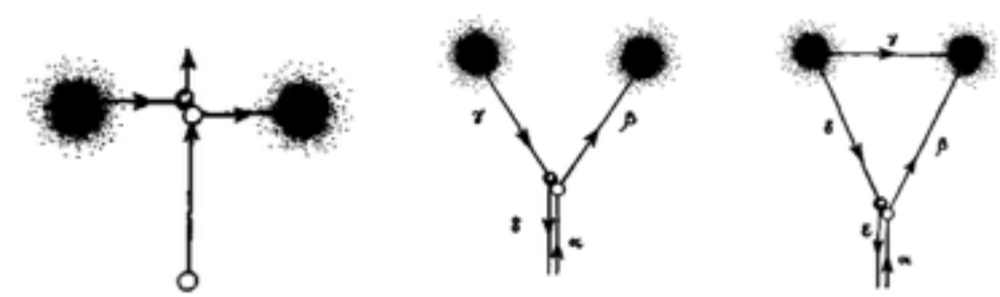
- Assumptions:
  Novikov's principle of self-consistency
  (no grandfather antinomy)

  No „new physics“ in the past



- Implications:

The Billiard Ball Crisis: An Infinity of Trajectories



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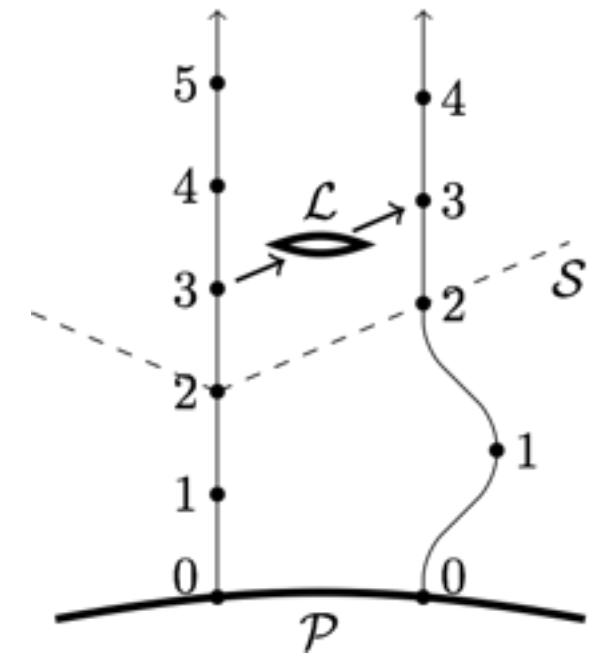
Time Travel



- Assumptions:
  Novikov's principle of self-consistency
  (no grandfather antinomy)

  No „new physics“ in local regions
  (not only at *P*)

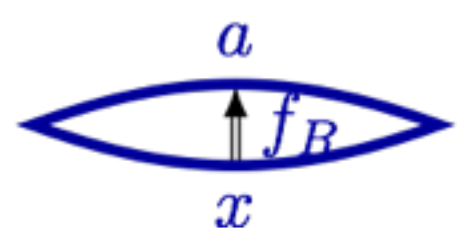
- Implications:
  Unique dynamics, reversibility, computationally tame

Ä. B., F. Costa, T. Ralph, S. Wolf, M. Zych, *arXiv:1703.00779 [gr-qc]* (2017)

Time Travel

- Local region R consists of *past* and *future* boundary

- Dynamics within R is described by a function f_R

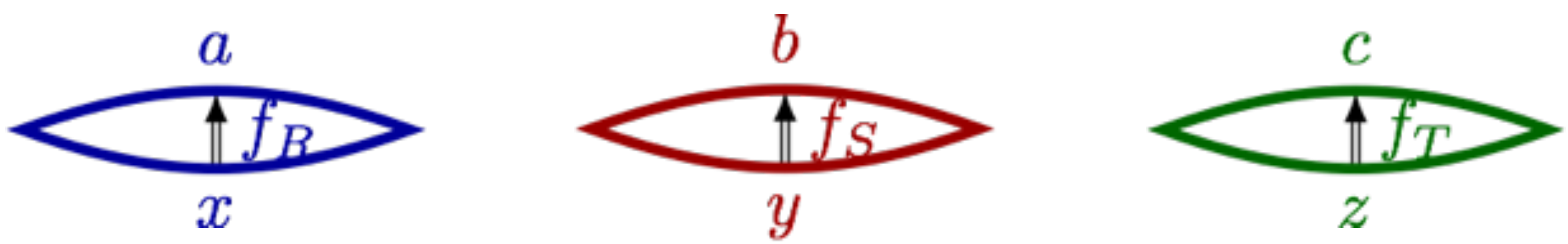


- No new physics: Any function f_R can be applied.

Ä. B., F. Costa, T. Ralph, S. Wolf, M. Zych, *arXiv:1703.00779 [gr-qc]* (2017)

Time Travel

- Multiple regions *R*,*S*,*T*



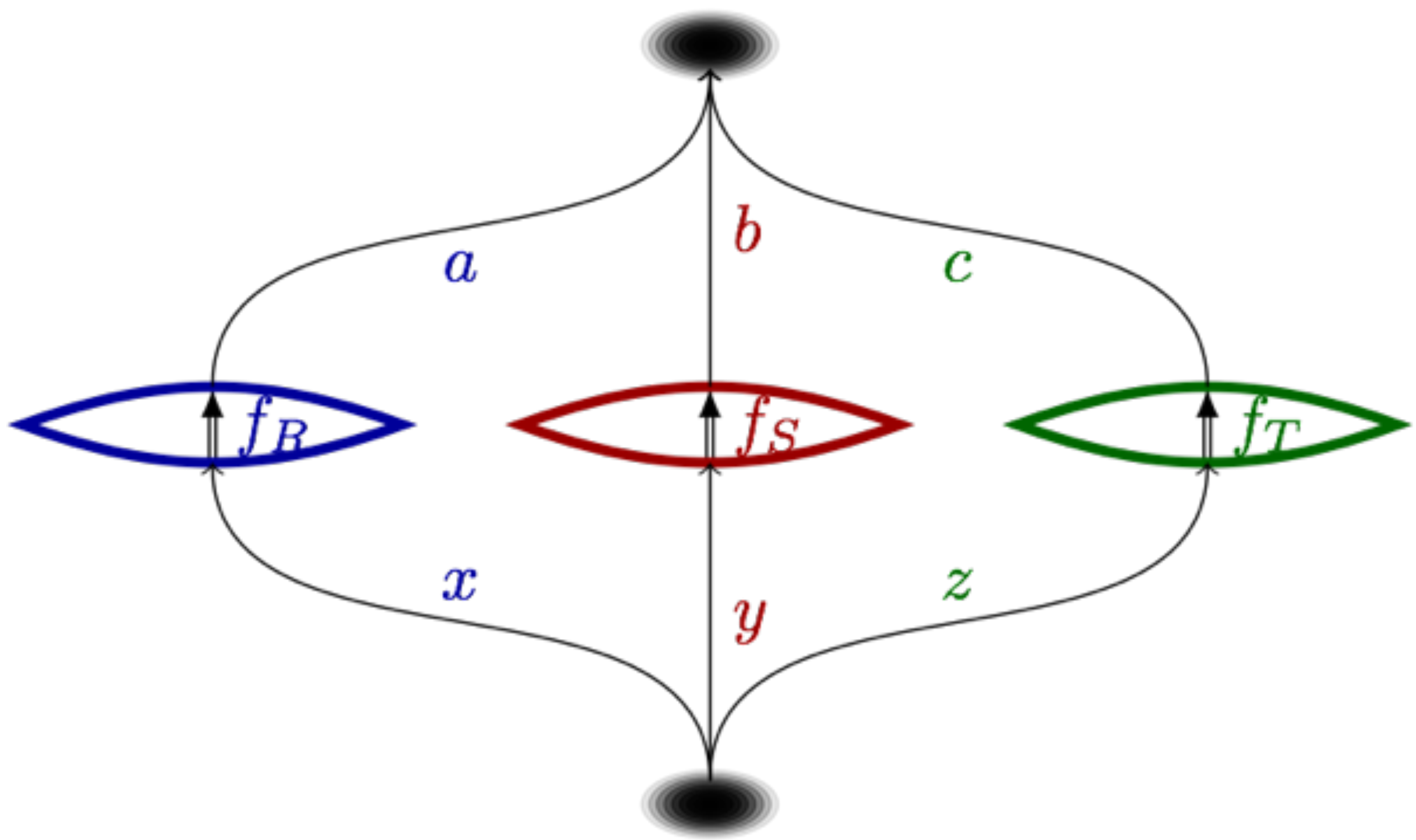
- Closed time-like curve as function

$$w : (a, b, c) \mapsto (x, y, z)$$

Ä. B., F. Costa, T. Ralph, S. Wolf, M. Zych, *arXiv:1703.00779 [gr-qc]* (2017)

Time Travel

- Closed time-like curve



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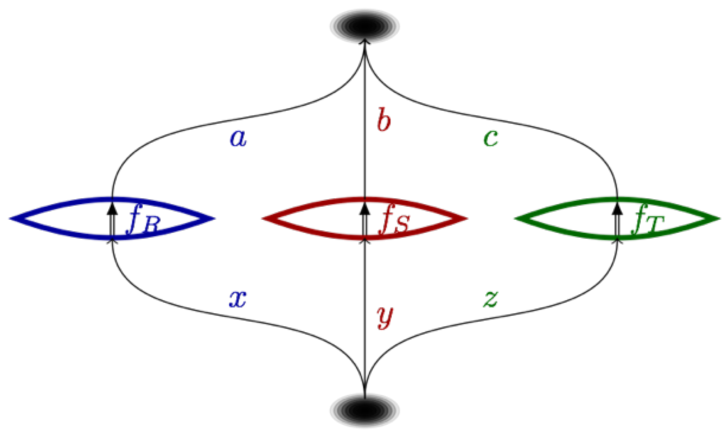
Ä. B., F. Costa, T. Ralph, S. Wolf, M. Zych, *arXiv:1703.00779 [gr-qc]* (2017)

Time Travel

- Novikov's self-consistency principle and **strong** „no new physics" principle

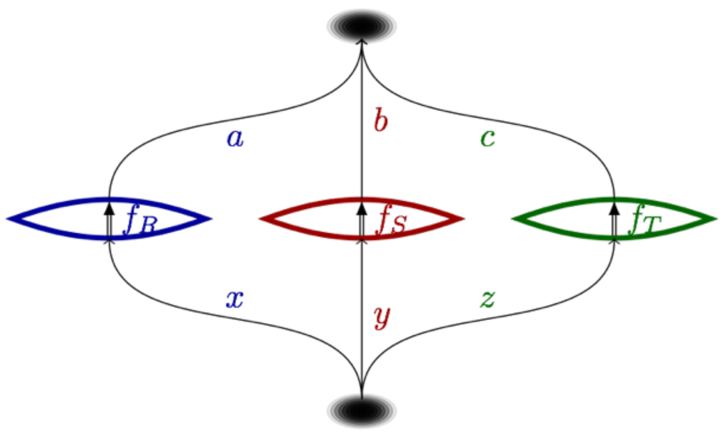


$$\forall f_R, f_S, f_T, \exists(x,y,z) : (x,y,z) = w(f_R(x), f_S(y), f_T(z))$$

Ä. B., F. Costa, T. Ralph, S. Wolf, M. Zych, *arXiv:1703.00779 [gr-qc]* (2017)

Time Travel

- Novikov's self-consistency principle and **strong** „no new physics" principle



$$\forall f_R, f_S, f_T, \exists(x, y, z) : (x, y, z) = w(f_R(x), f_S(y), f_T(z))$$

- This implies (unique dynamics, no information antinomy):

$$\forall f_R, f_S, f_T, \exists!(x, y, z) : (x, y, z) = w(f_R(x), f_S(y), f_T(z))$$

Ä. B., F. Costa, T. Ralph, S. Wolf, M. Zych, *arXiv:1703.00779 [gr-qc]* (2017)

Time Travel

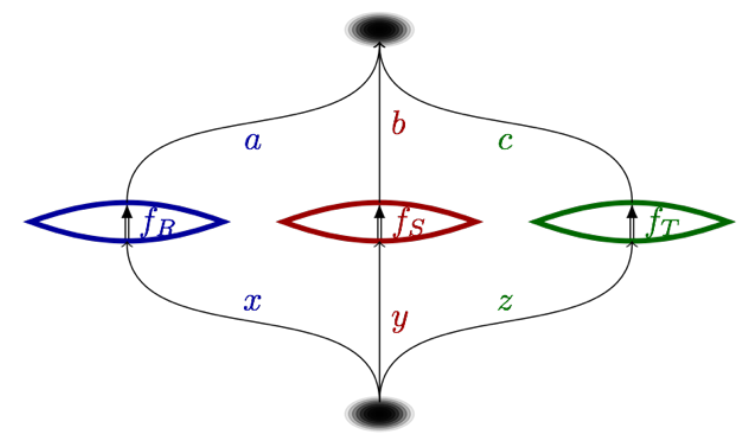
- Proof idea for:

$$\forall f_R, f_S, f_T, \exists!(x,y,z) : (x,y,z) = w(f_R(x), f_S(y), f_T(z))$$

N=1: Trivial
Induction: N -> N+1:
Assume *w* for N+1 regions has more than one fixed point.
Construct *w'* for N regions with more than one fixed point.

Ä. B., F. Costa, T. Ralph, S. Wolf, M. Zych, *arXiv:1703.00779 [gr-qc]* (2017)

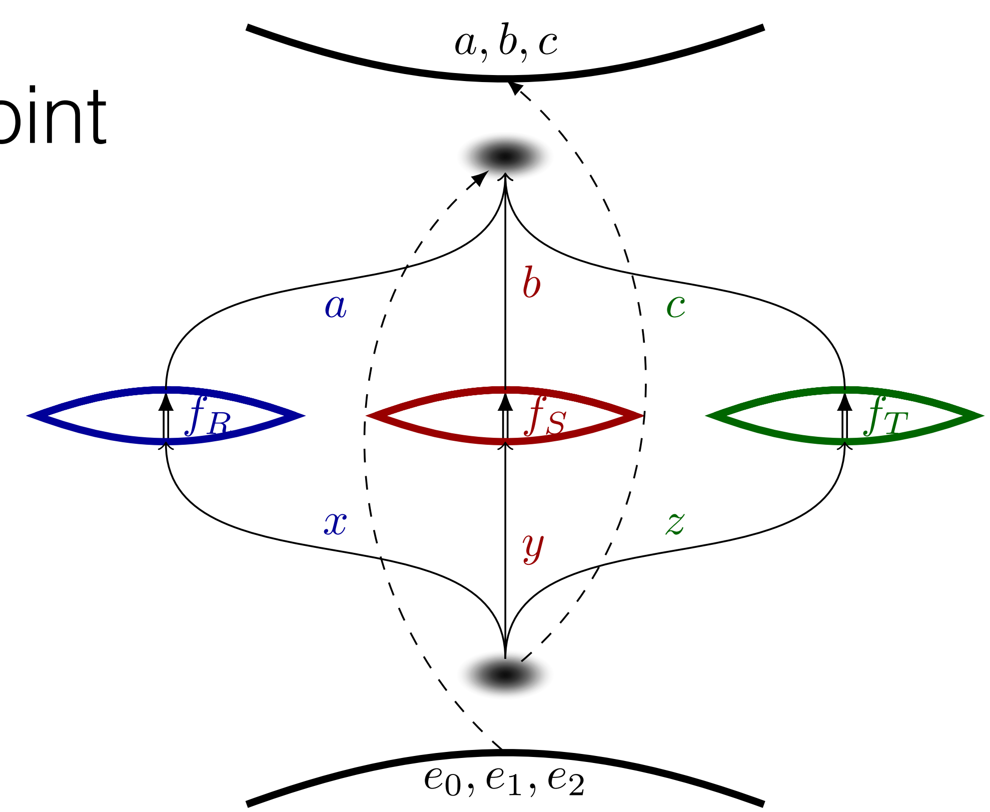
Time Travel



- Novikov's self-consistency principle and **strong** „no new physics" principle

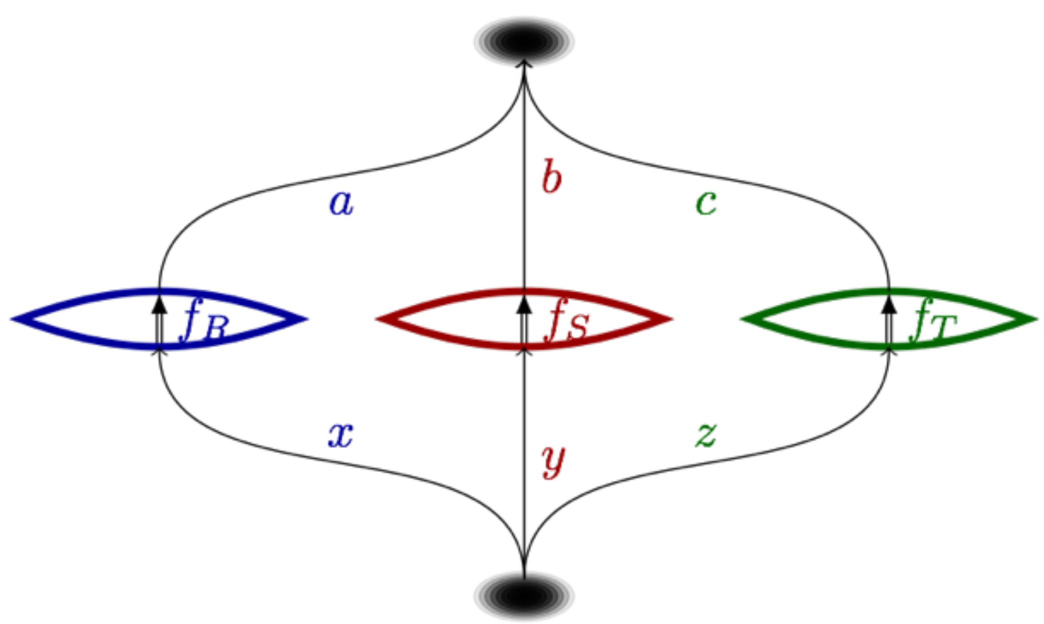
$$\forall f_R, f_S, f_T, \exists (x,y,z) : (x,y,z) = w(f_R(x), f_S(y), f_T(z))$$

- Every *w* that satisfies the fixed-point condition can be embedded in a reversible *w'* with two additional local regions



Ä. B., F. Costa, T. Ralph, S. Wolf, M. Zych, *arXiv:1703.00779 [gr-qc]* (2017)

Time Travel: Example



$$x = \neg b \wedge c$$
$$y = \neg c \wedge a$$
$$z = \neg a \wedge b$$

$$a = 0 \implies S \prec T$$
$$a = 1 \implies S \succ T$$

Ä. B., F. Costa, T. Ralph, S. Wolf, M. Zych, *arXiv:1703.00779 [gr-qc]* (2017)

Time Travel

Logically problematic?
- Grandfather antinomy
- Information antinomy

Computationally problematic?
- NP-Hardness assumption

Physically problematic?
- Reversibility of deterministic laws
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Conclusion

take home message

The logically consistent, classical world outside of the causal is

- *non empty*
- *computationally tame*
  (in the deterministic case; cannot efficiently solve NP-hard problems)
- *reversible with unique dynamics*
  (in the deterministic case)

thank you

Background image: M.C. Escher, *„Drawing Hands,"* (Lithograph, 1948)