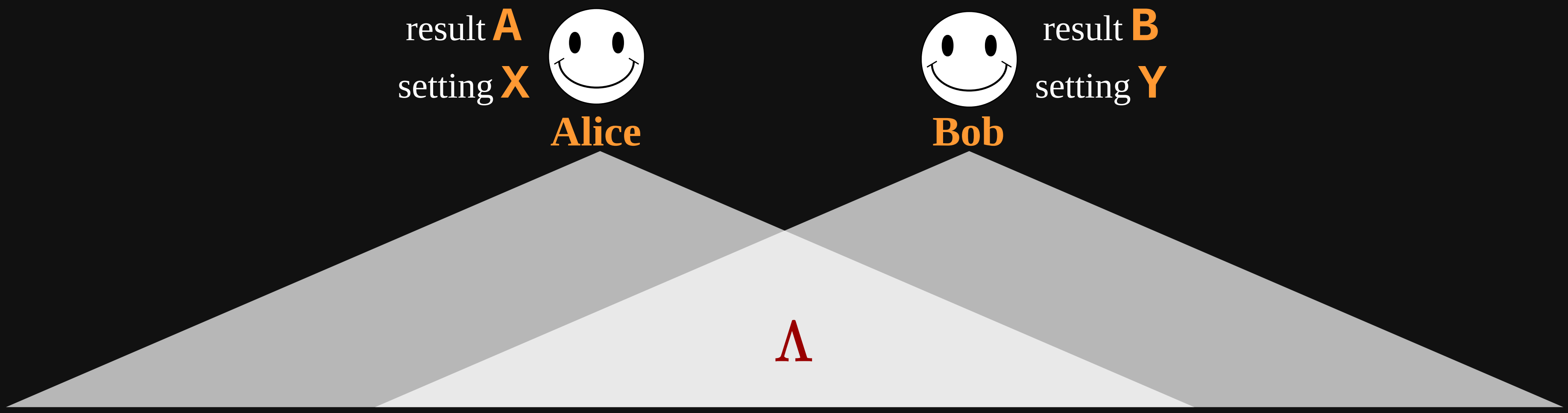
Graphical Tests of Causality

joint work with Eleftherios-Ermis Tselentis

Ämin Baumeler, Università della Svizzera italiana

June 4, 2025 @ **FoQaCiA** in Braga

PRA 111, 052211 (2025) *and ongoing…*



$$p(a,b|x,y) = \sum_{\lambda} p(\lambda)\ \ p(a|x,\lambda)\ \ p(b|y,\lambda)$$

$$\Pr[A \oplus B = X \cdot Y] \leq 3/4$$

$$p(a,b|x,y) = \sum_{\lambda} p(\lambda)\ p(a|x,\lambda)\ p(b|y,\lambda)$$

$$p(00|00) \quad p(01|00) \quad p(10|00) \quad p(11|00)$$
$$p(00|01) \quad p(01|01) \quad p(10|01) \quad p(11|01)$$
$$p(00|10) \quad p(01|10) \quad p(10|10) \quad p(11|10)$$
$$p(00|11) \quad p(01|11) \quad p(10|11) \quad p(11|11)$$

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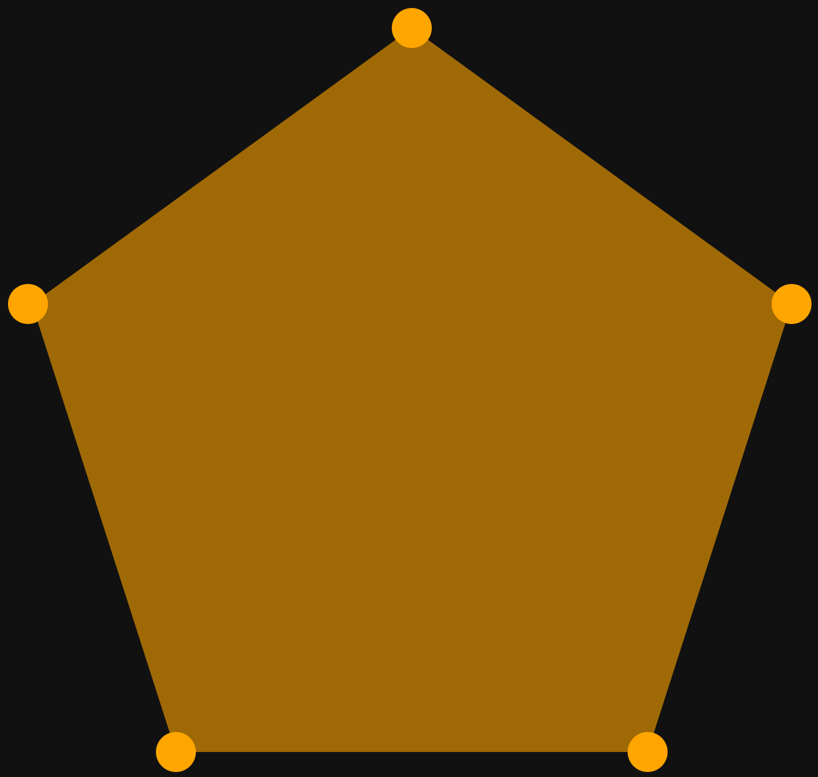
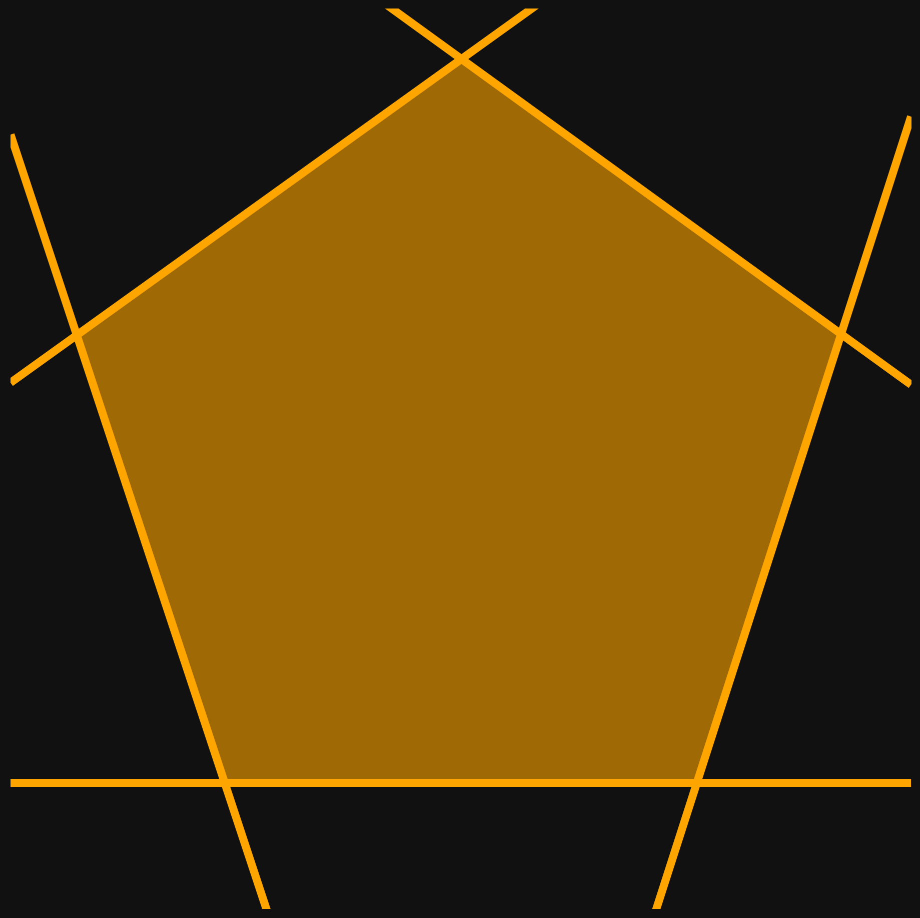
$$p(00|11) \quad p(01|11) \quad p(10|11)$$

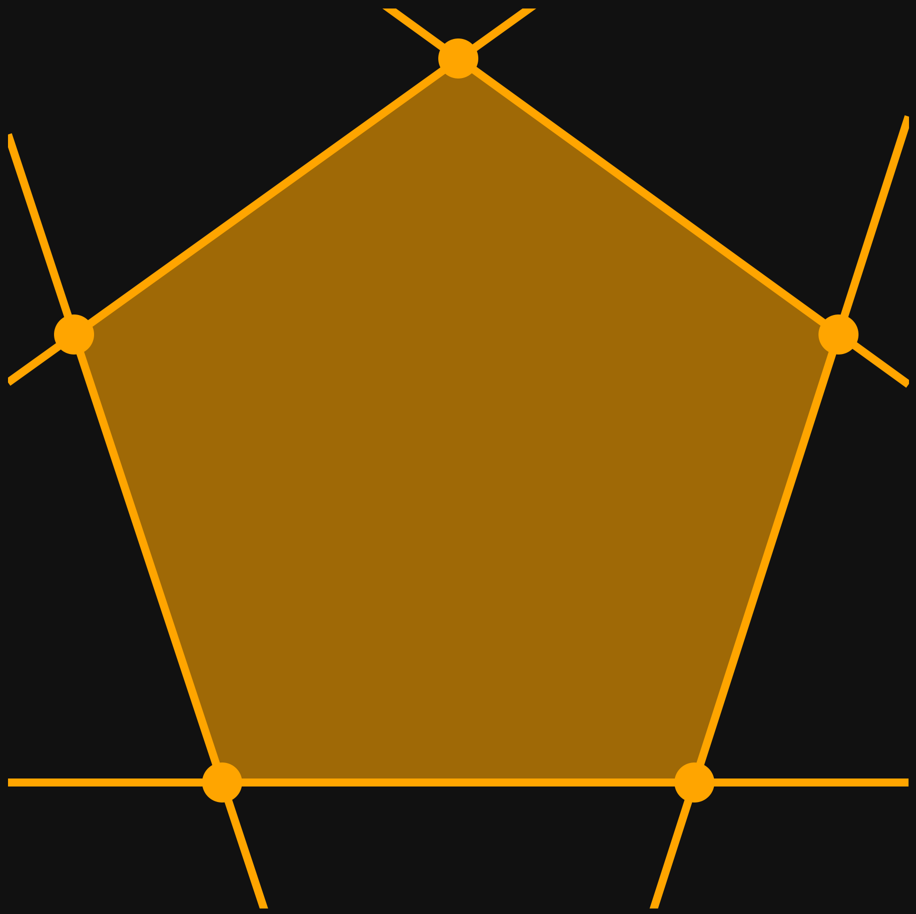
Geometric representation: convex polytope



vertex representation

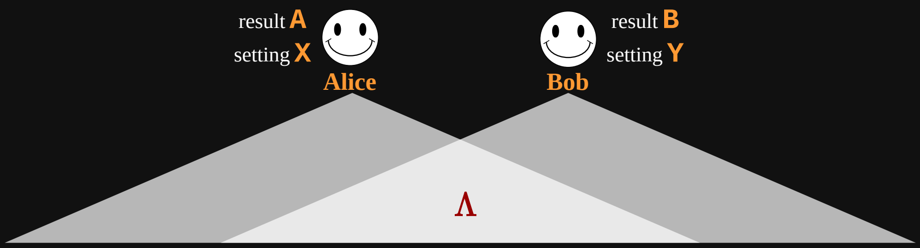


halfspace representation



Bell inequality defines **facet**

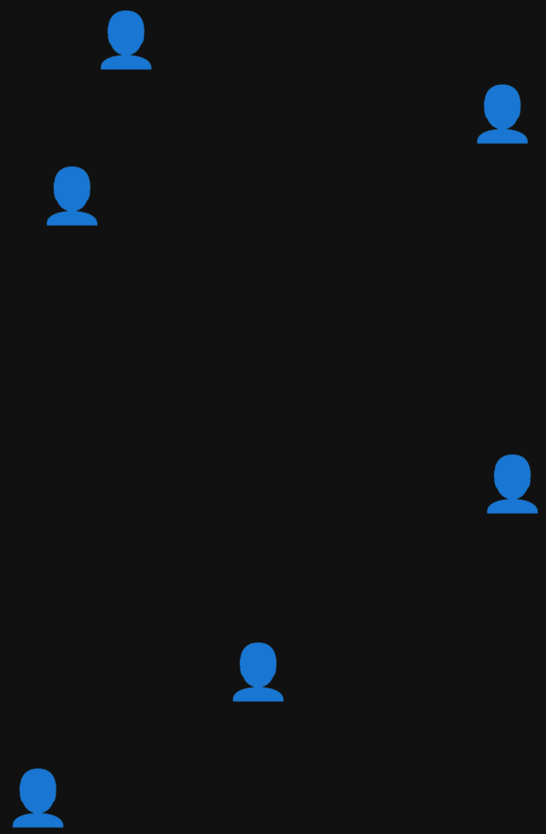
Outline



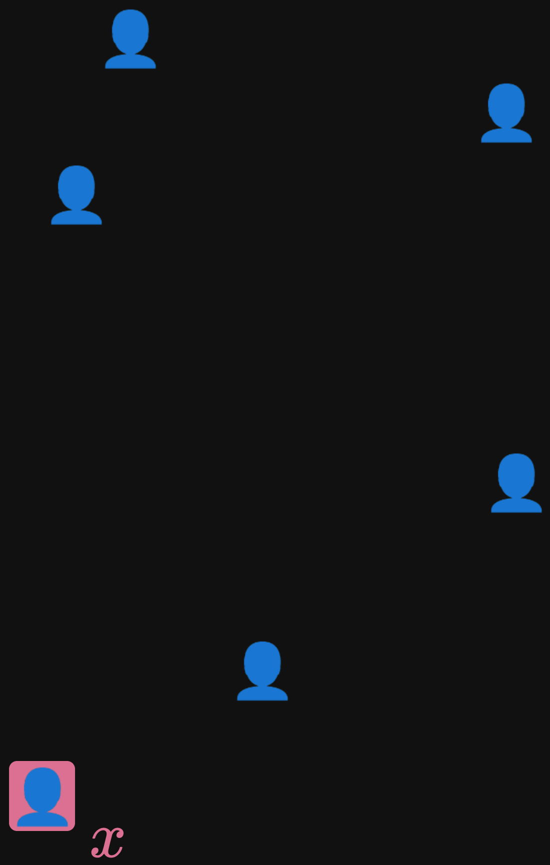
0. Bell Test of Locality
1. **Single-Output Scenario**
2. SOS Polytopes
3. Graphical Games
4. Notions of Causality
5. End

The Single-Output Scenario

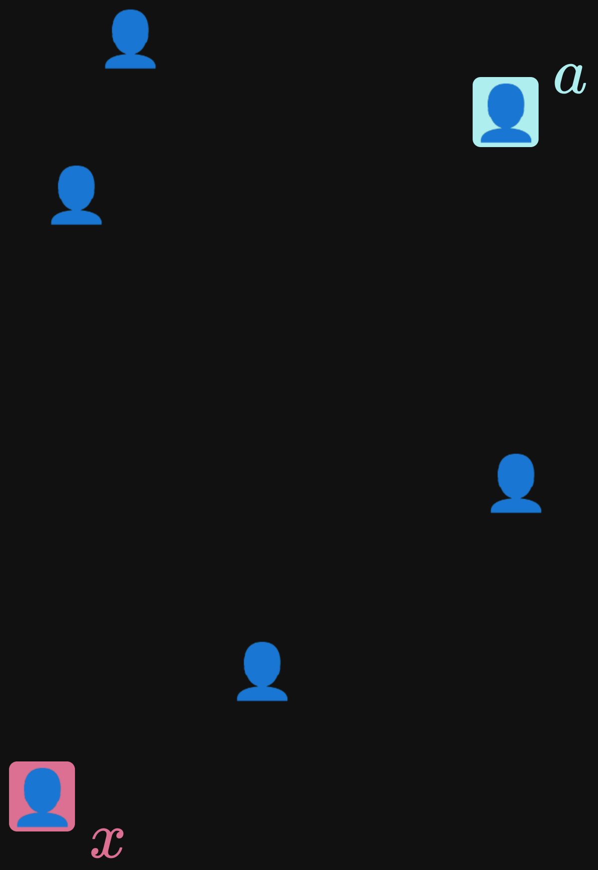
- n parties $\{0, 1, \ldots, n-1\}$



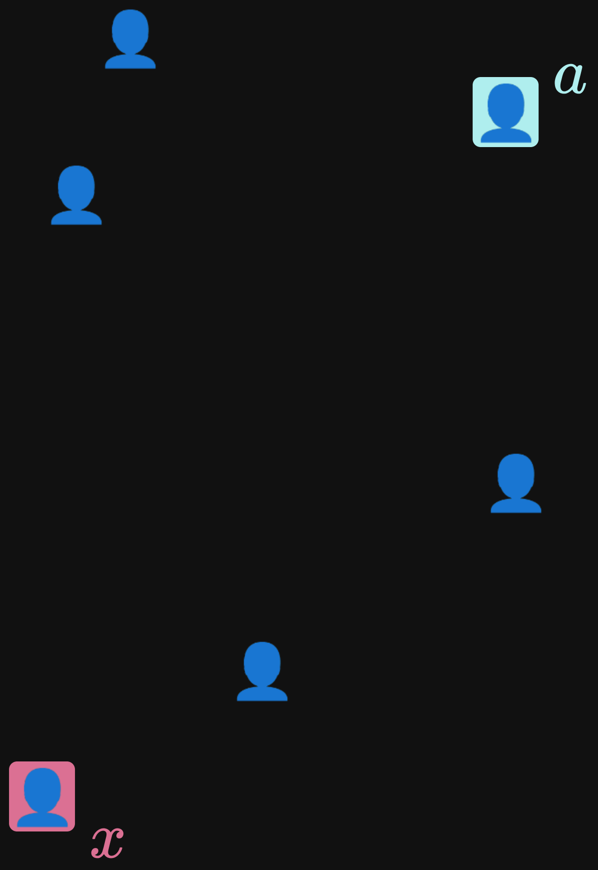
- n parties $\{0, 1, \ldots, n-1\}$
- party s with input x



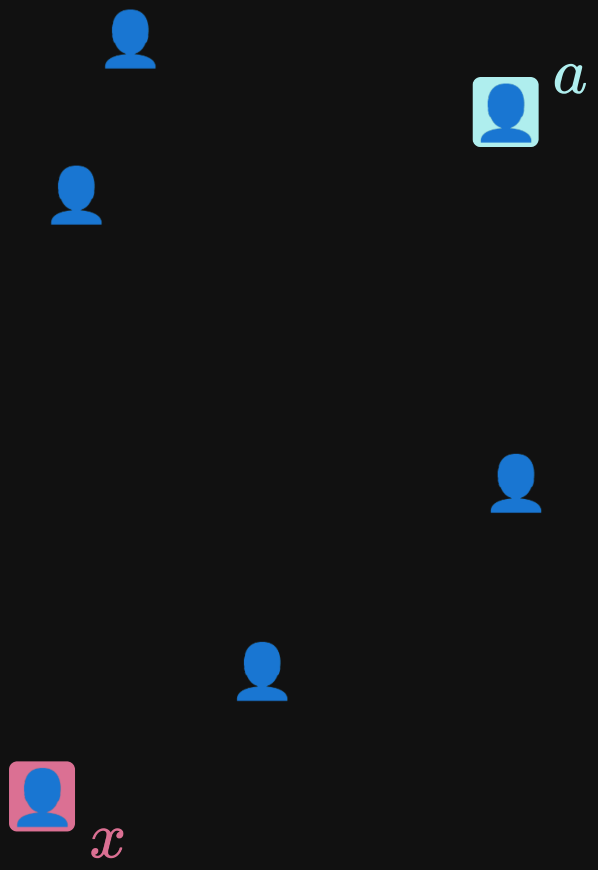
- n parties $\{0, 1, \ldots, n-1\}$
- party s with input x
- party r with output a



- n parties $\{0, 1, \ldots, n-1\}$
- party s with input x
- party r with output a
- $s \neq r$, x and a binary



- n parties $\{0, 1, \ldots, n-1\}$
- party s with input x
- party r with output a
- $s \neq r$, x and a binary
- all parties learn s and r



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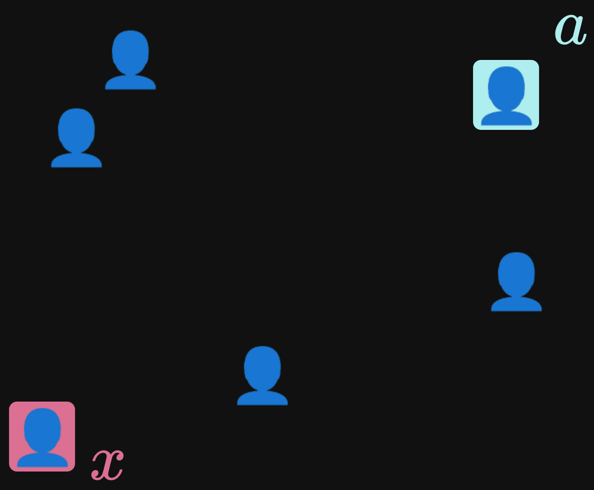
- $s \neq r$, x and a binary
- all parties learn s and r



$$p(a|s,r,x)$$

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$a \setminus s,r,x$	010	011	020	021	$\cdots$	100	101	120	121	$\cdots$
0	p^0_{010}	p^0_{011}	p^0_{020}	p^0_{021}		p^0_{100}	p^0_{101}	p^0_{120}	p^0_{121}	
1	p^1_{010}	p^1_{011}	p^1_{020}	p^1_{021}		p^1_{100}	p^1_{101}	p^1_{120}	p^1_{121}	

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0	p^0_{010}	p^0_{011}	p^0_{020}	p^0_{021}		p^0_{100}	p^0_{101}	p^0_{120}	p^0_{121}	

$$p(a = 0 | s, r, x)$$

$x \setminus s,r$	01	02	$\cdots$	10	12	$\cdots$
0	p^0_{010}	p^0_{020}		p^0_{100}	p^0_{120}	
1	p^0_{011}	p^0_{021}		p^0_{101}	p^0_{121}	

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0	p^0_{010}	p^0_{020}		p^0_{100}	p^0_{120}	
1	p^0_{011}	p^0_{021}		p^0_{101}	p^0_{121}	

$$d = 2n(n-1)$$

Outline

$x \setminus s,r$	01	02	$\cdots$	10	12	$\cdots$
0	p^0_{010}	p^0_{020}		p^0_{100}	p^0_{120}	
1	p^0_{011}	p^0_{021}		p^0_{101}	p^0_{121}	

0. Bell Test of Locality
1. Single-Output Scenario
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Polytope $\mathcal{P}$: General Properties

$x \setminus s,r$	01	02	$\cdots$	10	12	$\cdots$
0	0	1		1	0	
1	0	0		1	0	

Polytope $\mathcal{P}$: General Properties

Suppose $ext(\mathcal{P}) \subseteq \{0,1\}^d$

$x \setminus s,r$	01	02	$\cdots$	10	12	$\cdots$
0	0	1		1	0	
1	0	0		1	0	

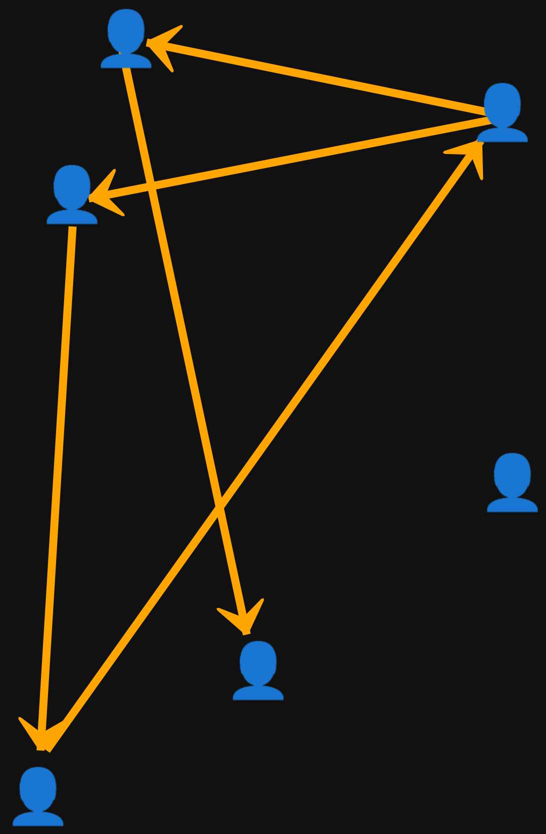
$x \setminus s,r$	01	02	$\cdots$	10	12	$\cdots$
0	0	1		1	0	
1	0	0		1	0	

$x \setminus s,r$	01	02	$\cdots$	10	12	$\cdots$
0	0	$1 \oplus 1$		1	0	
1	0	$0 \oplus 1$		1	0	

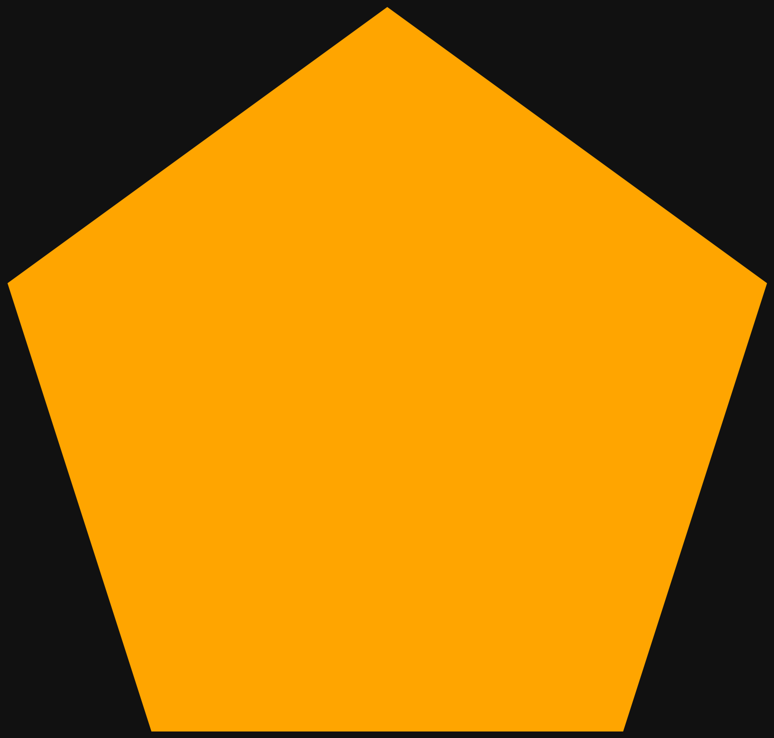
Equivalence relation

$x \setminus s,r$	01	02	$\cdots$	10	12	$\cdots$
0	0	1		1	0	
1	0	0		1	0	
$\oplus$	0	1		0	0	

$$\{q \mid q \sim p, p, q \in ext(\mathcal{P})\}:$$



Projection of $\mathcal{P}$ to the polytope $\mathcal{Q}$ of directed graphs



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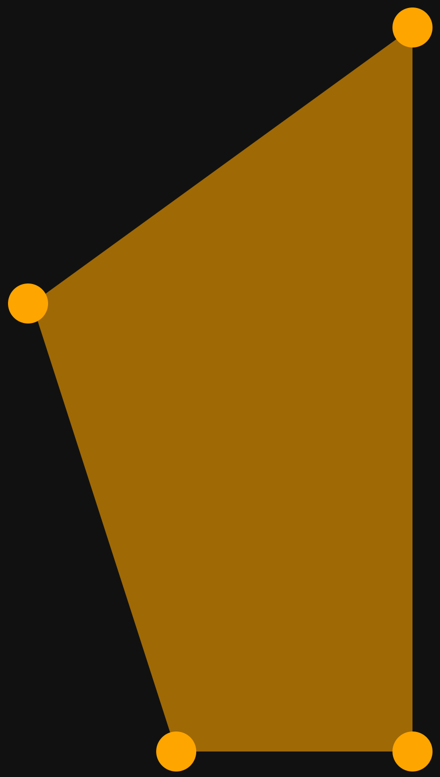

Lifting Lemma

If $\mathcal{P}$ is full-dimensional and deterministic, then every nontrivial and nonnegative facet-defining inequality of the graph polytope $\mathcal{Q}$ describes a facet-defining inequality of $\mathcal{P}$.

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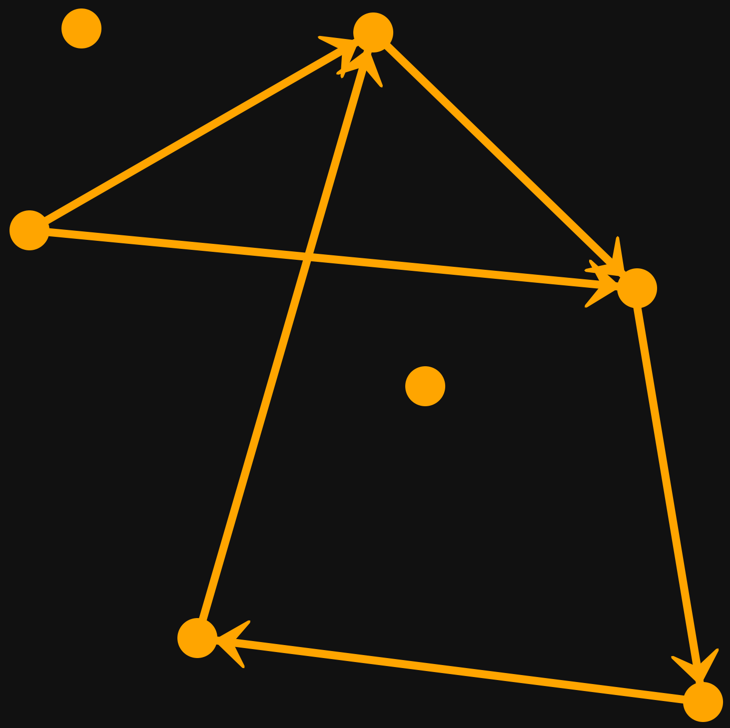
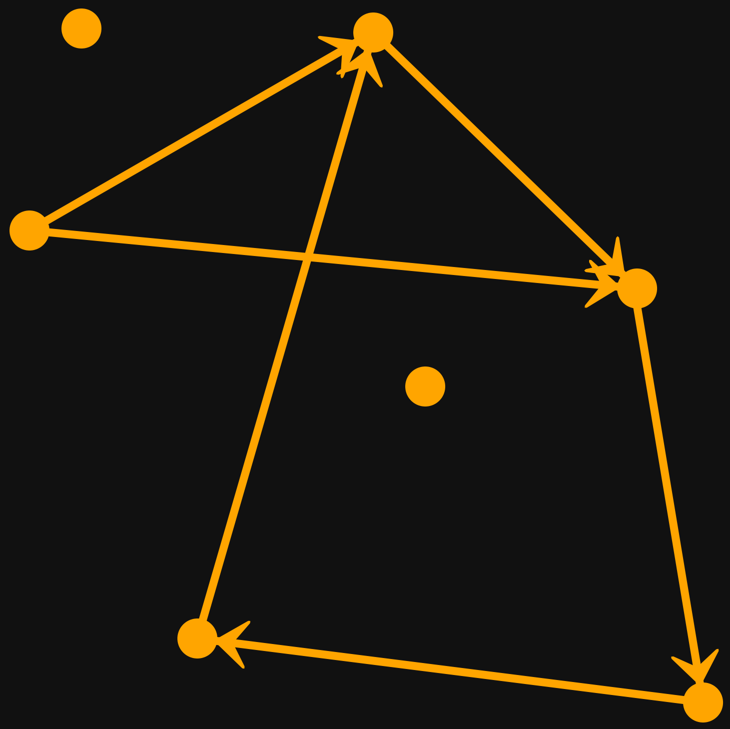
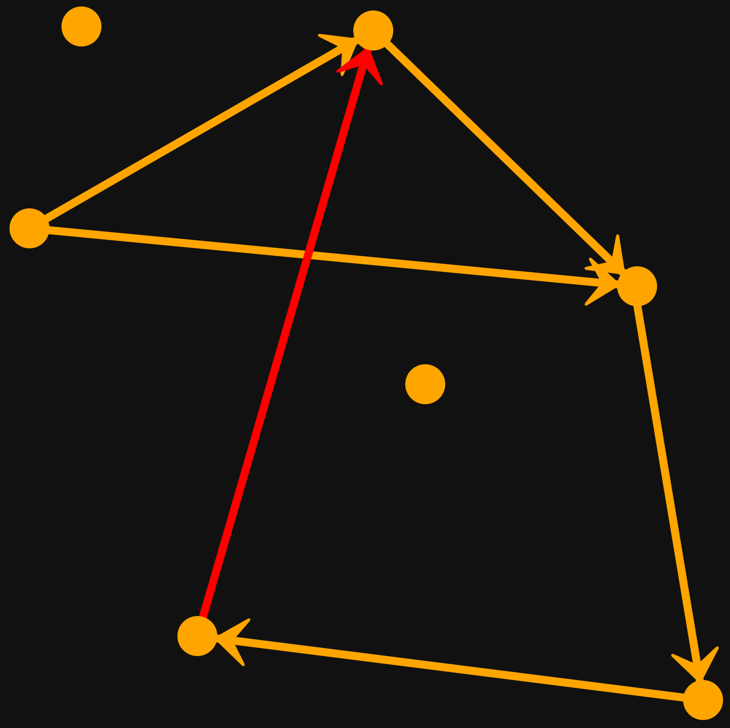
Graphical Games

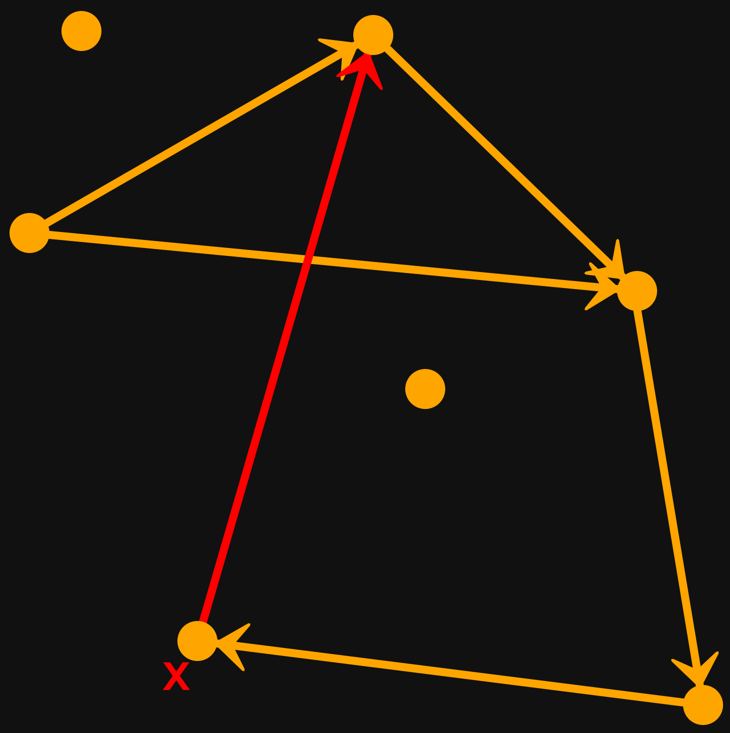




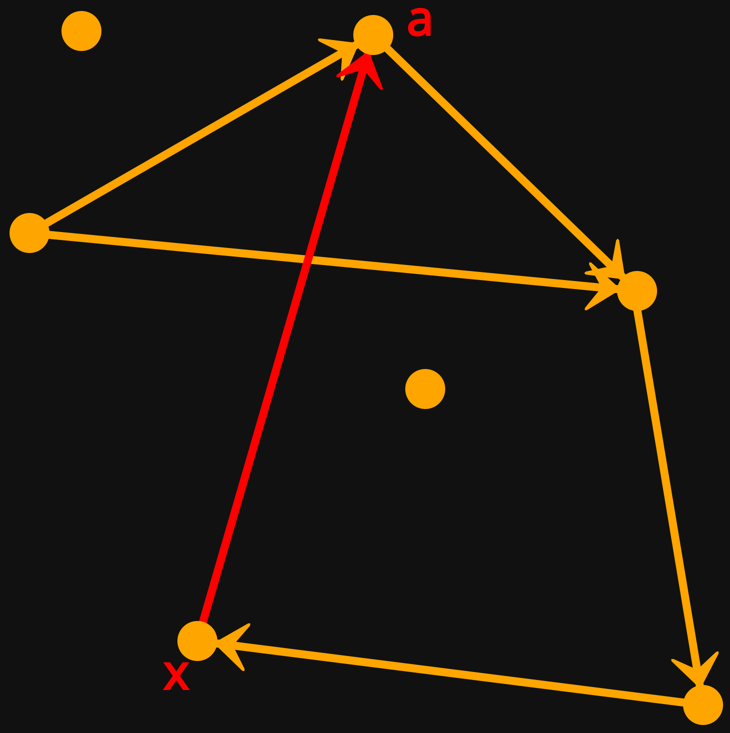
Game $\mathcal{G}(D)$ from directed graph $D = (V(D), E(D))$



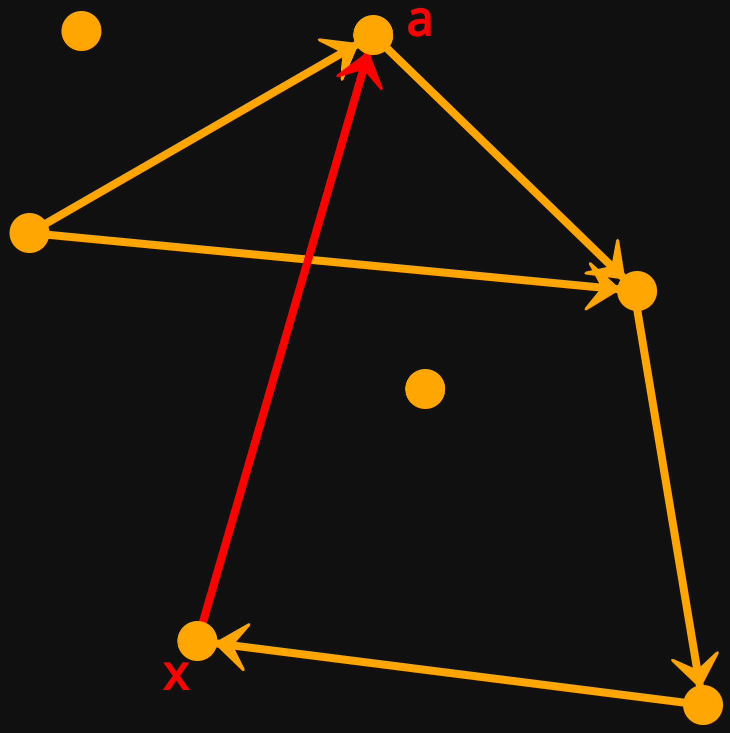
Referee broadcasts $(s \to r) \in_R E(D)$



... announces $x \in_R \{0, 1\}$ to r



Party r produces a guess $a \in \{0,1\}$



The parties **win** $\mathcal{G}(D)$ whenever $a = x$.

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Notions of Causality

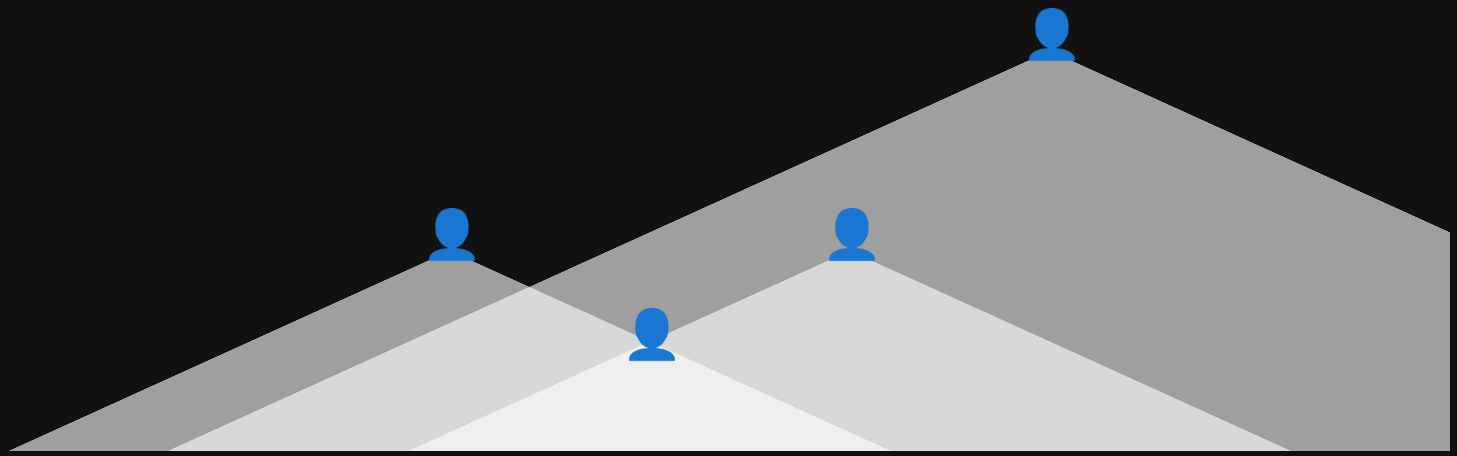
Notions of Causality

- Static causal order
- Dynamic causal order
- Bicausal
- ...

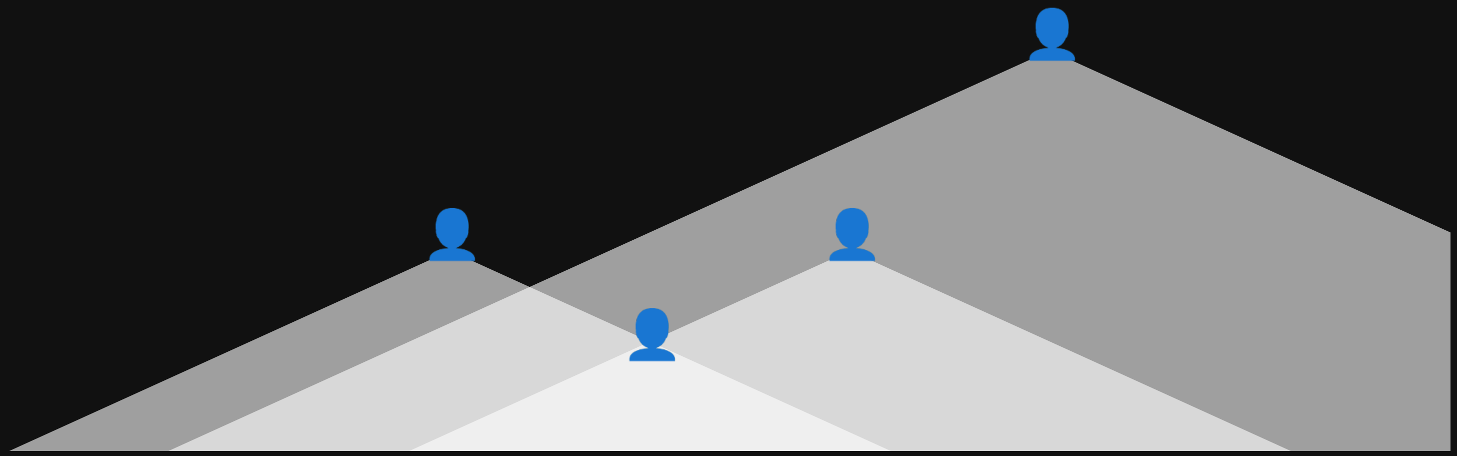
Notions of Causality

- Static causal order (SR)
- Dynamic causal order (GR)
- Bicausal
- ...

Static Causal Order



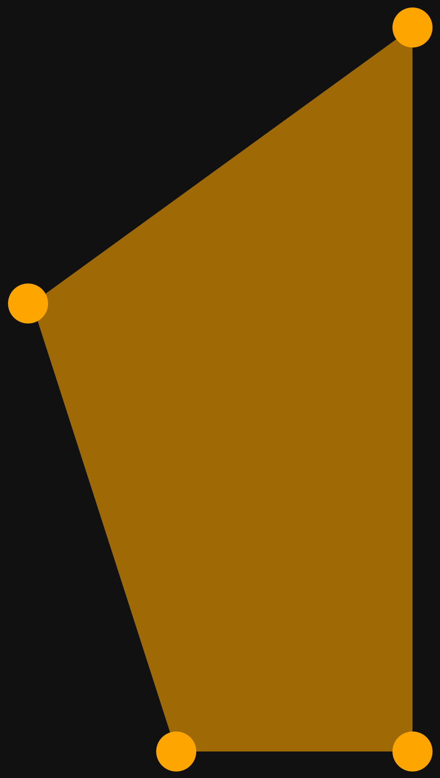
Static Causal Order



$$p(a|s,r,x) =$$

$$\sum_{\sigma:s\not\preceq_\sigma r} p(\sigma)p_\sigma^{\not\preceq}(a|s,r) + \sum_{\sigma:s\preceq_\sigma r} p(\sigma)p_\sigma^{\preceq}(a|s,r,x)$$

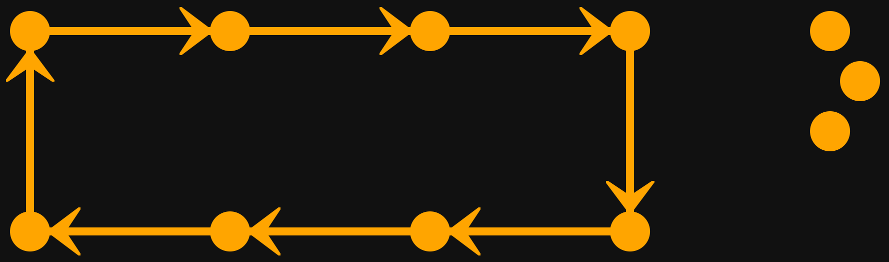
Static Causal Order: the DAG polytope



Static Causal Order: the DAG polytope

- k-cycle inequalities
- k-fence inequalities
- k-Möbius inequalities
- ...

k-cycle inequalities

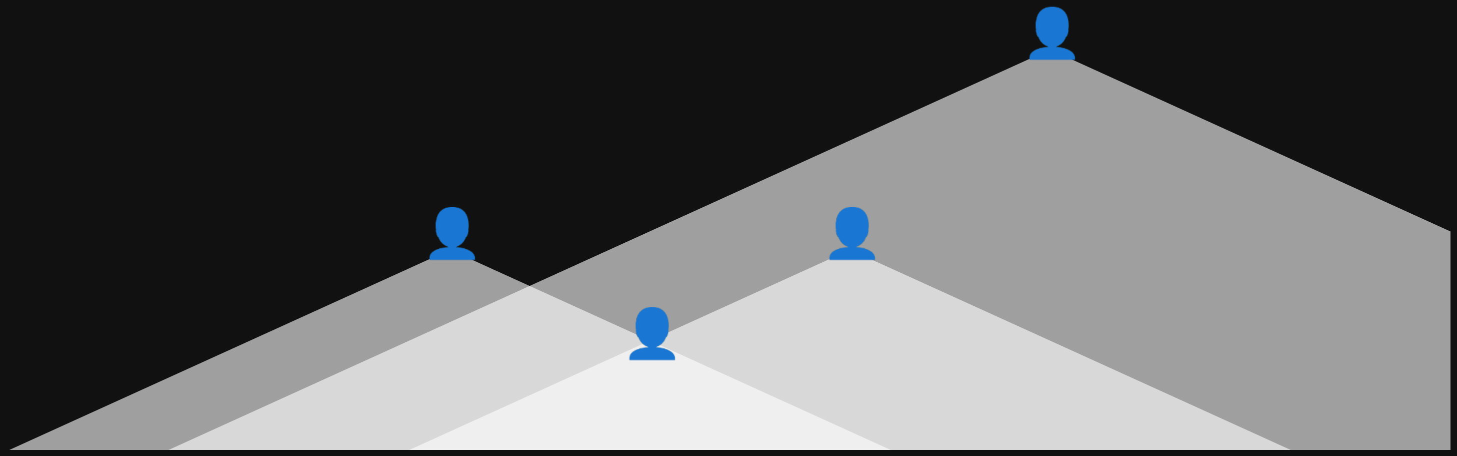


$$\Pr[\text{win } \mathcal{G}(D)] \leq 1 - \frac{1}{2k}$$

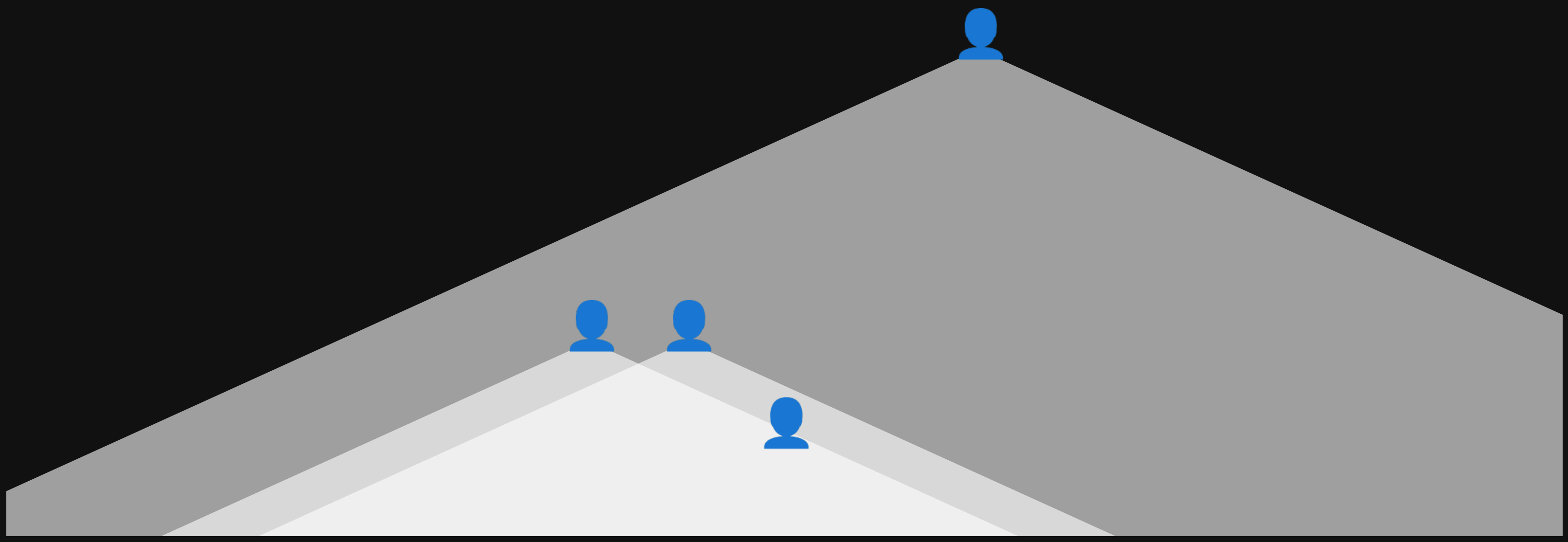
k-Möbius inequalities

$$\Pr[\text{win } \mathcal{G}(D)] \leq 1 - \frac{k+1}{12k} < \frac{11}{12}$$

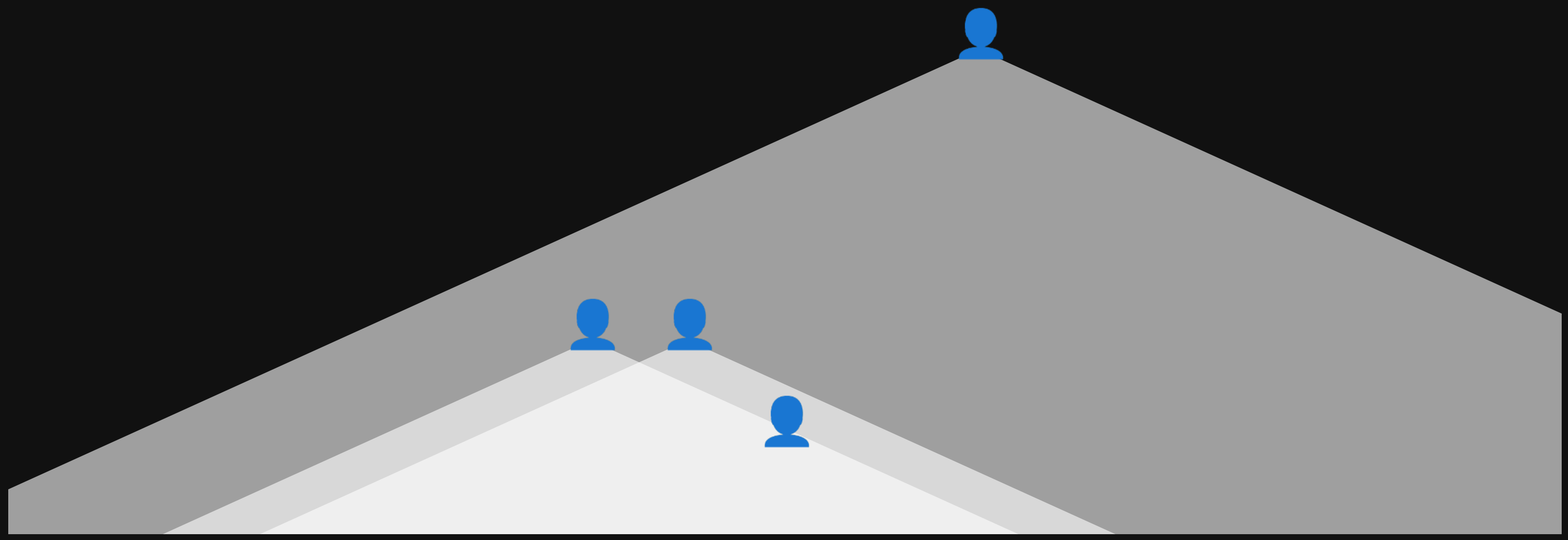
Dynamic Causal Order



Dynamic Causal Order



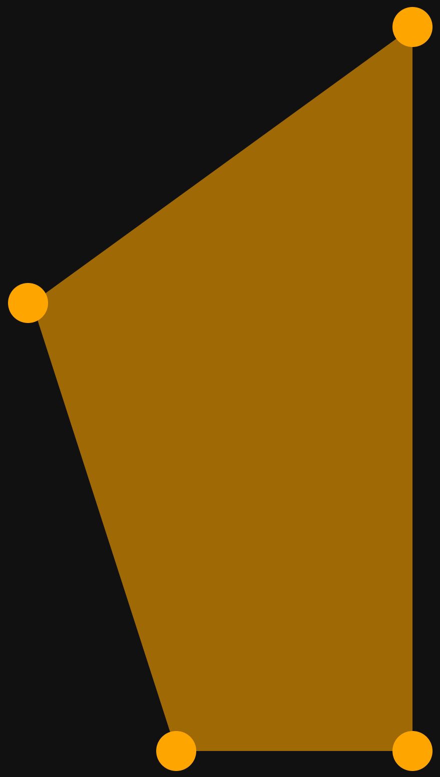
Dynamic Causal Order



$$p(a|s, r, x) =$$

$$p_{\text{first}}(r)\ p_0(a|s, r) + (1 - p_{\text{first}}(r))\ p_1(a|s, r, x)$$

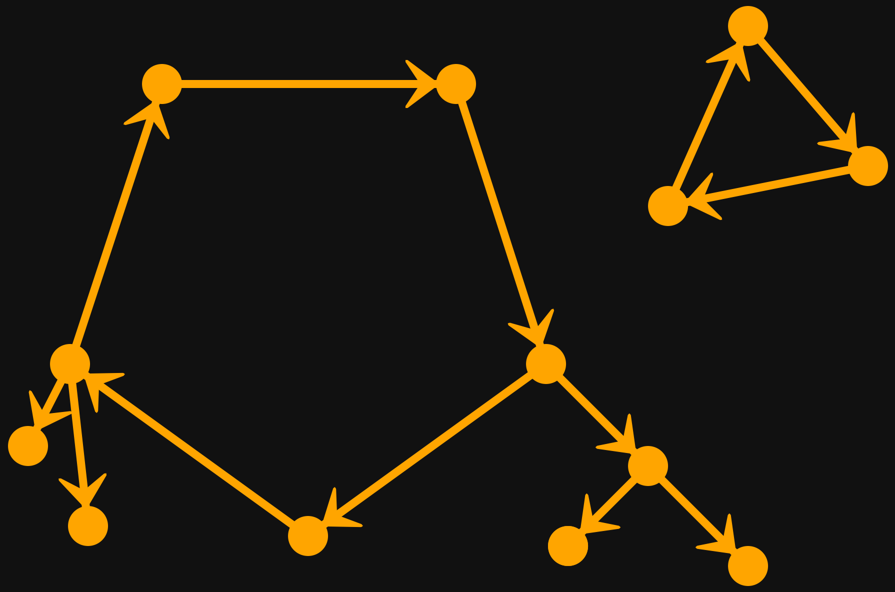
Dynamic Causal Order: the source graph polytope



Dynamic Causal Order: the source graph polytope

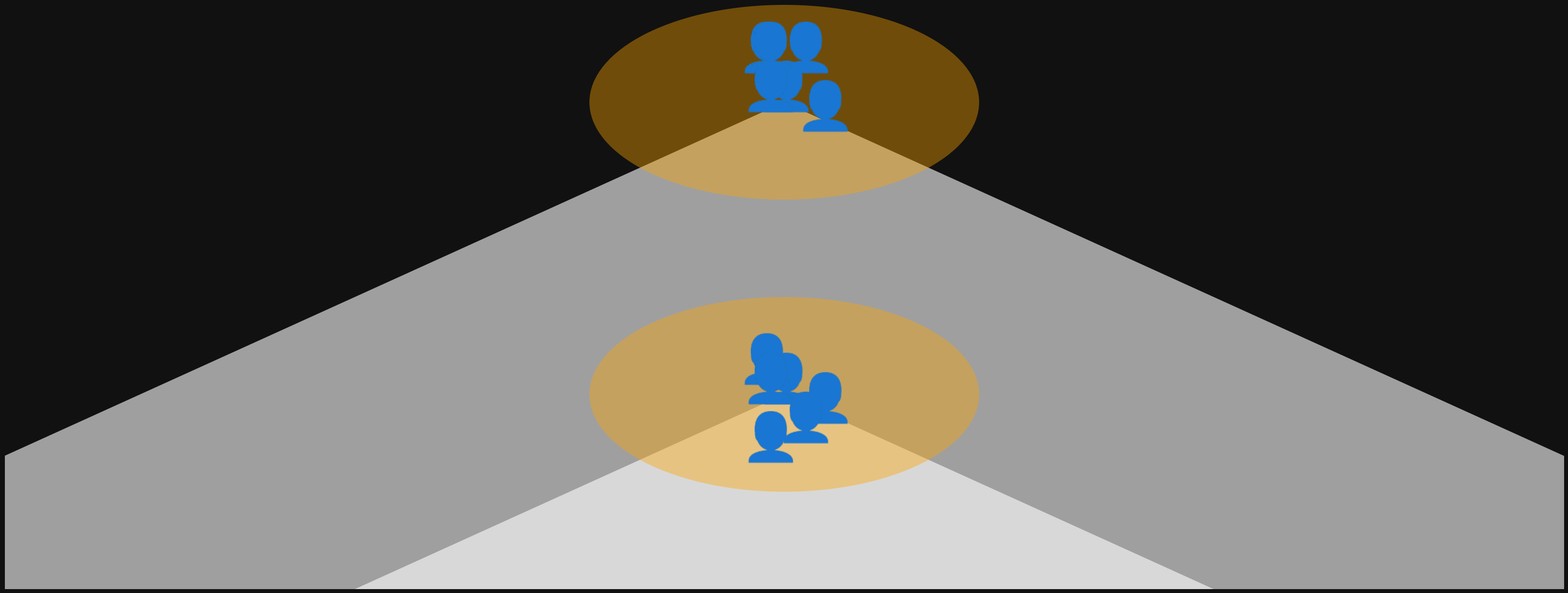
- Kefalopoda inequalities
- (more?)

Kefalopoda inequalities

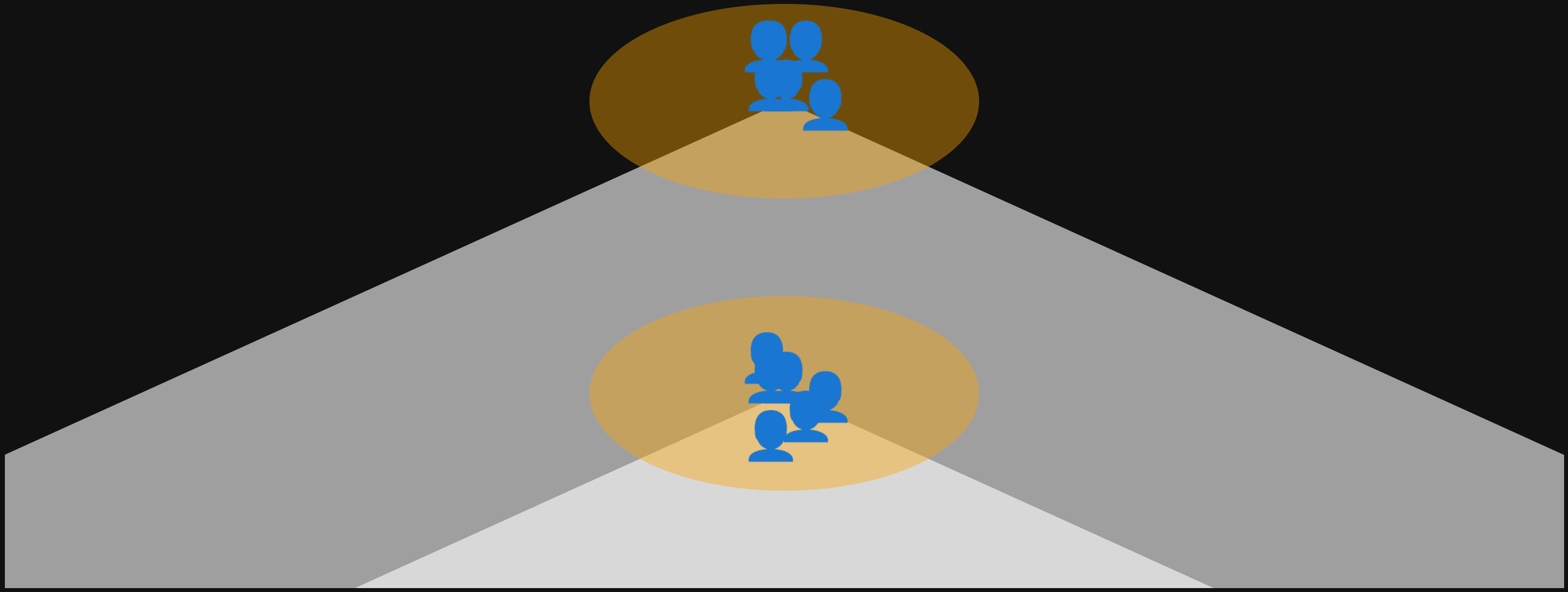


$$\Pr[\text{win } \mathcal{G}(D)] \leq 1 - \frac{1}{n}$$

Bicausal



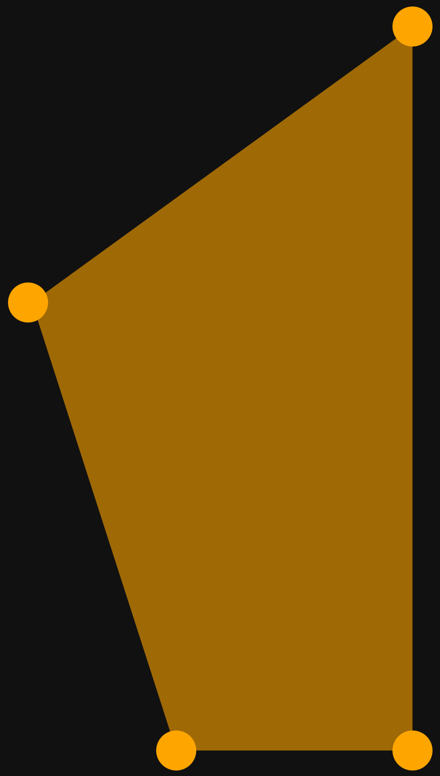
Bicausal



$$p(a|s,r,x) =$$

$$\sum_{\mathcal{F}:s\notin\mathcal{F}\wedge r\in\mathcal{F}} p(\mathcal{F})\, p_0(a|s,r) + \sum_{\mathcal{F}:r\in\mathcal{F}\rightarrow s\in\mathcal{F}} p(\mathcal{F})\, p_1(a|s$$

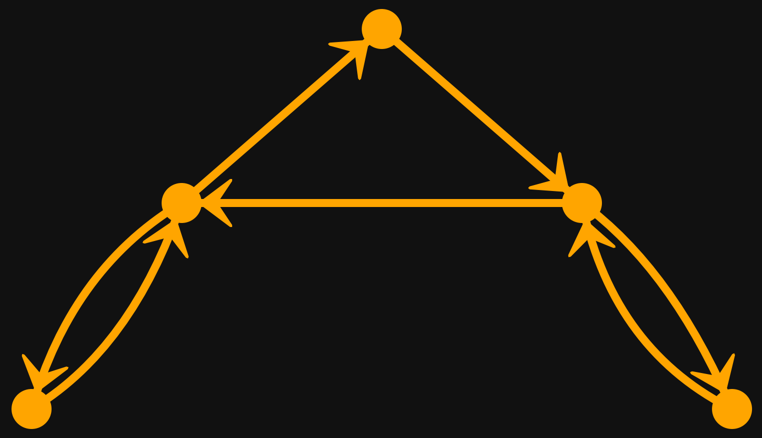
Bicausal: the not-strong polytope



Bicausal: the not-strong polytope

- Minimally strong
- (more!)

Minimally strong inequalities



$$\Pr[\text{win } \mathcal{G}(D)] \leq 1 - \frac{1}{|E(D)|}$$

Thank You