Admissible Causal Structures and Violations of Causal Inequalities

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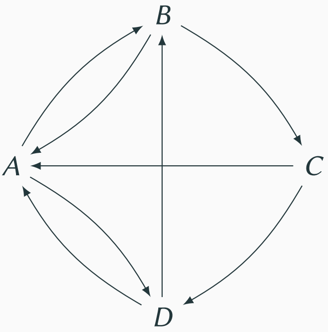
IQOQI-Vienna, Austrian Academy of Sciences

What are the *qualitative* limitations on

- causal structures and
- correlations

imposed by *local quantum mechanics?*



Motivations

Information Processing

Novel forms of communication, *e.g.,* local operation and classical *cyclic* communication (extension of LOCC, *cf.* R. Kunjwal, ÄB, arXiv:2202.00440).

Classical vs. Quantum

Does quantum theory allow for more general causal structures?

Testing Quantum Gravity

Exceeding the limits implies incompatibility with local quantum mechanics.

Higher-Order Computation

Cast quantum maps as higher-order maps (*cf.* poster by L. Apadula).

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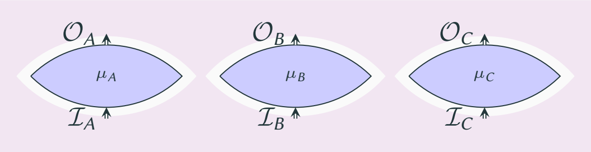
Outline

Preliminaries: Processes and Causal Models

Processes

Quantum Process W

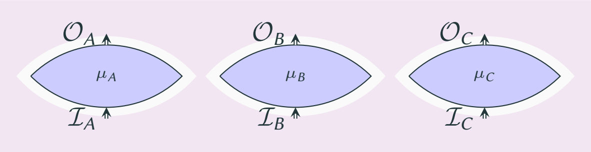
- $W \in \mathcal{L}(\bigotimes_k \mathcal{I}_k \otimes O_k)$ positive semi-definite,
- $\forall \{\mu_k \in \mathrm{CPTP}(\mathcal{I}_k, \mathcal{O}_k)\}_k : \mathrm{Tr}\left[W\left(\bigotimes_k \rho^{\mu_k}\right)\right] = 1.$



O. Oreshkov, F. Costa, Č. Brukner, Nat. Comm 3 (2012)

Classical-Deterministic Process ω

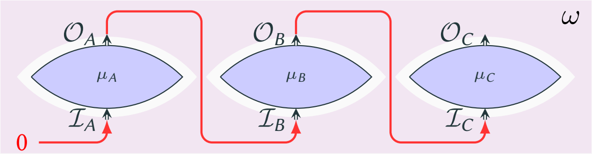
- $\omega : \bigtimes_k \mathcal{O}_k \to \bigtimes_k \mathcal{I}_k$,
- $\forall \{\mu_k : \mathcal{I}_k \to \mathcal{O}_k\}_k \exists r : r = \omega(\mu(r))$.



ÄB, S. Wolf, NJP 18 (2016)

Processes: Example

$$\omega : \{0,1\}^3 \to \{0,1\}^3$$
$$(o_A, o_B, o_C) \mapsto (0, o_A, o_B)$$



Split-Node Causal Model

A **split-node** *causal model* consists of

- a causal structure (directed ~~*acyclic*~~ graph, $G = (V, E)$)
  where V is a set of *parties* (each has input, output space),
- model parameters $\{\rho_{k|\mathsf{Pa}(k)}\}$.

Classical Deterministic

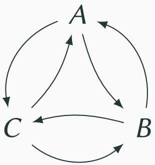
- $\rho_{k|\mathsf{Pa}(k)} : \mathcal{O}_{\mathsf{Pa}(k)} \to \mathcal{I}_k$
- $\omega := (\rho_{k|\mathsf{Pa}(k)})_k$

Quantum

- $\rho_{k|\mathsf{Pa}(k)}$: Choi(CPTP($\mathcal{O}_{\mathsf{Pa}(k)}, \mathcal{I}_k$))
  with $[\rho_{i|\mathsf{Pa}(i)}, \rho_{j|\mathsf{Pa}(j)}] = 0$
- $W = \prod_k \rho_{k|\mathsf{Pa}(k)}$

J. Barrett, R. Lorenz, O. Oreshkov, Nat Comm 12 (2021)

Split-Node Causal Model: Example



$$\mathcal{I}_k = \mathcal{O}_k = \{0, 1\}$$

$$\begin{aligned} \big\{ \rho_{A|B,C} &: \mathcal{O}_B \times \mathcal{O}_C \to \mathcal{I}_A, \\ \rho_{B|A,C} &: \mathcal{O}_A \times \mathcal{O}_C \to \mathcal{I}_B, \\ \rho_{C|A,B} &: \mathcal{O}_A \times \mathcal{O}_B \to \mathcal{I}_C \big\} \end{aligned}$$

$$\omega := (\rho_{A|B,C}, \rho_{B|A,C}, \rho_{C|A,B})$$

Split-Node Causal Model

A **split-node** *causal model* is

- **faithful** iff an edge $u \to v$ implies that party u *can* signal to party v
- **consistent** iff W/ω is a quantum/classical-deterministic process

J. Barrett, R. Lorenz, O. Oreshkov, Nat Comm 12 (2021)

Admissible Causal Structures

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A causal structure (directed graph) $G = (V, E)$ is **admissible** if and only if there exists a *faithful* and *consistent* split-node causal model with causal structure G.

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Questions:

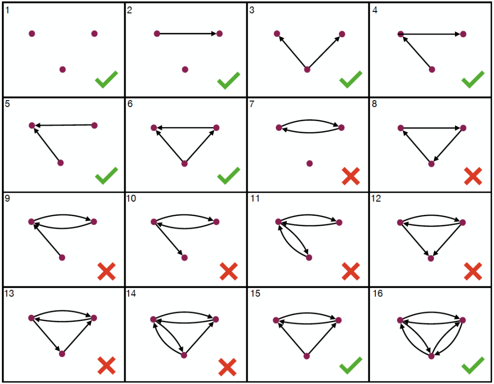
- Which causal structures are (in)admissible?
- Does there exist a causal structure that is admissible in the quantum, but inadmissible in the classical-deterministic case?

Admissible Causal Structures

Distinct nodes i, j in a graph are **siblings** iff $|\mathrm{Pa}(i) \cap \mathrm{Pa}(j)| > 0$.

Definition (SOC: Siblings-On-Cycles Graph)

We call a directed graph $G = (V, E)$ a **siblings-on-cycles graphs** (SOC) if and only if each directed cycle in G has siblings.



Admissible Causal Structures: Inadmissibility

Statement (Inadmissibility)

If $G = (V, E)$ *is* NOT *a SOC, then G is* inadmissible*.*

This holds in the classical-deterministic *and* in the quantum case.

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Statement (Inadmissibility)

If $G = (V, E)$ is NOT *a SOC, then G is* inadmissible.

This holds in the classical-deterministic *and* in the quantum case.

Proof sketch (quantum):

- If a path $u \to n_1 \to \cdots \to n_k \to v$ has no siblings, then there exist interventions such that party u can signal to party v.
- G contains a cycle $u \to \cdots \to u$ *without* siblings, so party u can signal to her/his own past.

Admissible Causal Structures: Admissibility

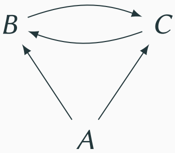
Conjecture (Classical-deterministic process)

Given a SOC $G = (V, E)$ the following faithful *classical-deterministic causal model is consistent:*

- $\mathcal{I}_k = \{0, 1\}$
- $\mathcal{O}_k = \mathsf{Ch}(k) \cup \{\perp\}$
- $$\begin{aligned}\rho_{k|\mathsf{Pa}(k)} : \mathcal{O}_{\mathsf{Pa}(k)} &\to \mathcal{I}_k \\ (o_j)_{j \in \mathsf{Pa}(k)} &\mapsto \prod_{j \in \mathsf{Pa}(k)} [k = o_j]\end{aligned}$$

Admissible Causal Structures: Admissibility

Example:



$$\mathcal{I}_A = \mathcal{I}_B = \mathcal{I}_C = \{0, 1\}$$

$$\mathcal{O}_A = \{B, C, \bot\}$$

$$\mathcal{O}_B = \{C, \bot\}$$

$$\mathcal{O}_C = \{B, \bot\}$$

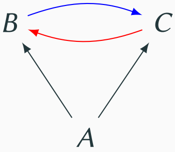
$$\rho_{A|\emptyset}(\emptyset) = 1$$

$$\rho_{B|A,C}(o_A, o_C) = [B = o_A][B = o_C]$$

$$\rho_{C|A,B}(o_A, o_B) = [C = o_A][C = o_B]$$

Admissible Causal Structures: Admissibility

Example:



$$\mathcal{I}_A = \mathcal{I}_B = \mathcal{I}_C = \{0, 1\} \qquad \rho_{A|\emptyset}(\emptyset) = 1$$

$$\mathcal{O}_A = \{B, C, \perp\} \qquad \rho_{B|A,C}(o_A, o_C) = [B = o_A][B = o_C]$$

$$\mathcal{O}_B = \{C, \perp\} \qquad \rho_{C|A,B}(o_A, o_B) = [C = o_A][C = o_B]$$

$$\mathcal{O}_C = \{B, \perp\}$$

$$o_A = B : \rho_{B|A,C}(B, o_c) = [B = o_c], \qquad \rho_{C|A,B}(B, o_B) = 0$$

$$o_A = C : \rho_{C|A,B}(C, o_B) = [C = o_B], \qquad \rho_{B|A,C}(C, o_c) = 0$$

Intuition:

The construction effectively breaks the directed cycle by control via parent:

Admissible Causal Structures: Admissibility

If the conjecture holds:

Statement (Admissibility)

If $G = (V, E)$ *is a SOC, then* G *is* admissible.

Admissible Causal Structures: Admissibility

If the conjecture holds:

Statement (Admissibility and Inadmissibility)

The graph $G = (V, E)$ is a SOC if and only if G is admissible.

Admissible Causal Structures: Admissibility

If the conjecture holds:

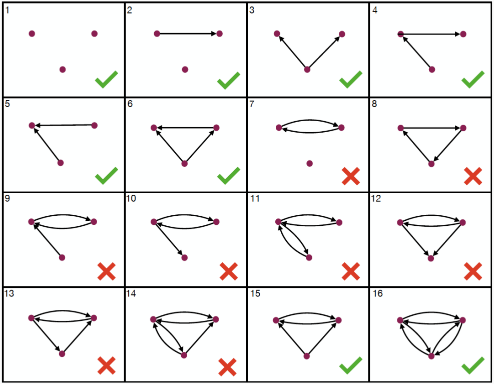
Statement (Admissibility and Inadmissibility)

The graph $G = (V, E)$ is a SOC if and only if G is admissible.

Corollary

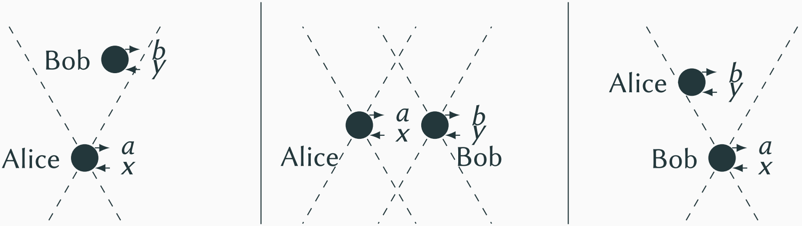
The set of admissible causal structures in the quantum and in the classical-deterministic case coïncide.

Admissible Causal Structures



Correlations

Causal Correlations



$$p(a, b \mid x, y) = q \times p(a \mid x)p(b \mid a, x, y) + (1 - q) \times p(b \mid y)p(a \mid b, x, y)$$

O. Oreshkov, F. Costa, Č. Brukner, Nat. Comm 3 (2012)

Causal Correlations

Definition (Causal Correlations)

For a set V of parties, the correlations $p(a_V \mid x_V)$ are causal if and only if

$$p(a_V \mid x_V) = \sum_k q_k p(a_k \mid x_k)\, p_{(a_k,x_k)}(a_{V\setminus\{k\}} \mid x_{V\setminus\{k\}}) \,,$$

where $p_{(a_k,x_k)}(a_{V\setminus\{k\}} \mid x_{V\setminus\{k\}})$ are *causal* correlations.

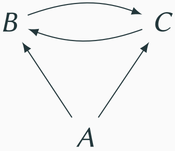
O. Oreshkov, C. Giarmatzi, NJP 18 (2016); A. Abbott, C. Giarmatzi, F. Costa, C. Branciard, PRA 94 (2016)

Causal Structures and Correlations

Questions:

- Which causal structures *do not exhibit* non-causal correlations?
- Which causal structures *do exhibit* non-causal correlations?

Causal Structures and Correlations: Causal Correlations



For a directed graph G, a cycle C is called **induced** if and only if the subgraph $G[C]$ is the cycle graph.

Statement (Causal Correlations)

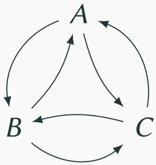
Let ω be a classical-deterministic process.
If all cycles in the causal structure of ω are ***induced****, then ω yields only causal correlations.*

Causal Structures and Correlations: Causal Correlations

Proof sketch:

- Lemma: A graph where all cycles are induced has a *source* node
- Reduce over source node, and repeat.
- This gives a decomposition of the correlation of the form

$$p(a_V \mid x_V) = \sum_k q_k p(a_k \mid x_k)\, p_{(a_k,x_k)}(a_{V\setminus\{k\}} \mid x_{V\setminus\{k\}})$$



Statement (Non-Causal Correlations)

Let ω be a classical-deterministic process.

If the causal structure of ω contains a cycle C where all common parents are inside C, i.e.,

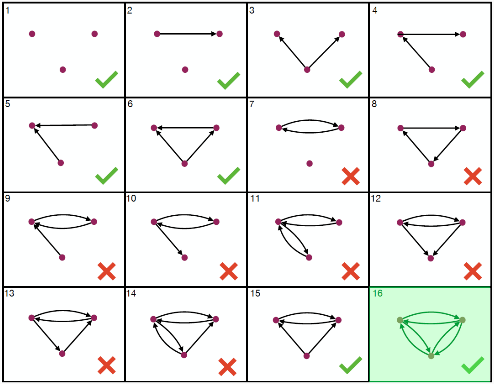
$$\bigcup_{k \neq \ell \in C} \mathsf{Pa}(k) \cap \mathsf{Pa}(\ell) \subseteq C$$

then ω yields non-causal correlations.

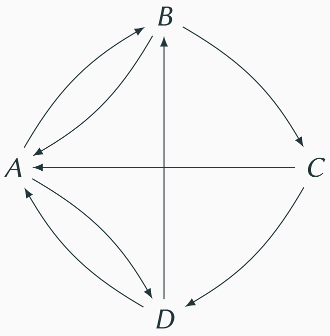
Proof sketch:

- Consider the following game played by all parties in C:
    1. A random party $k \in C$ is selected
    2. A random bit b is distributed to all parties in $C \setminus \{k\}$
    3. The parties win the game whenever party k guesses b correctly
- With causal correlations, this game is won with probability at most $1 - 1/2|C|$
- The process ω allows for a *deterministic violation* of this inequality

Causal Structures and Correlations



Summary

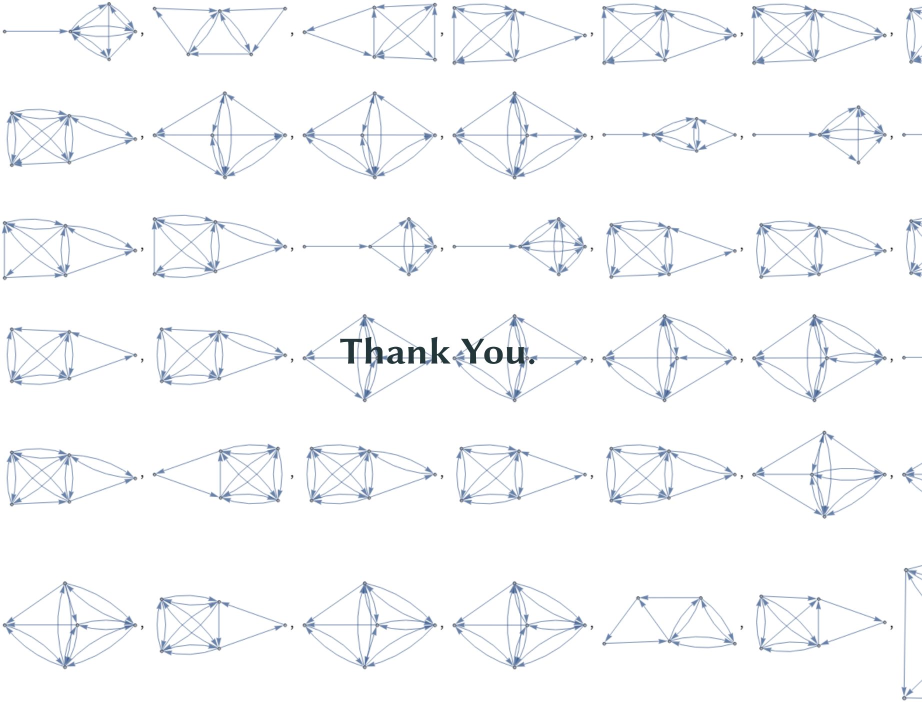


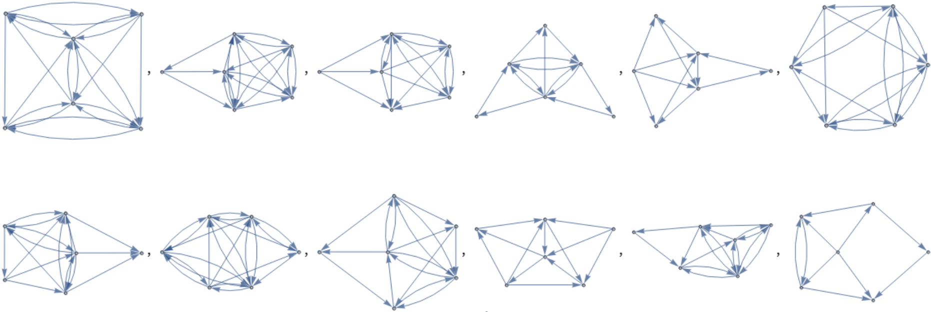
Cycles:
$(AB), (ABC), (ABCD)$
$(AD), (ADB), (ADBC)$
(BCD)

Induced cycles:
$(AB), (AD), (BCD)$

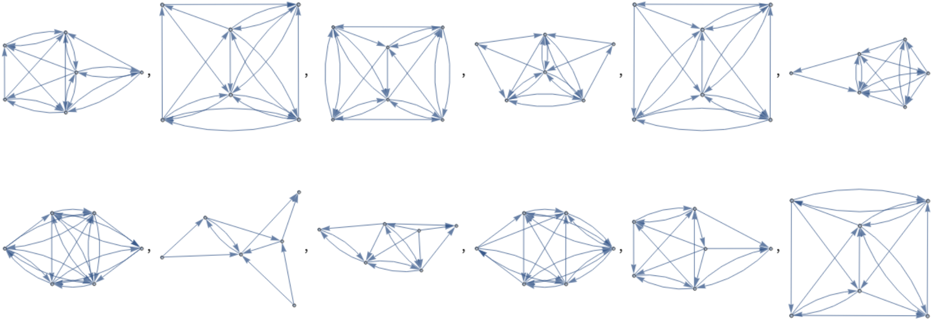
Violation of causal inequality:
$\{A, B, C, D\}$

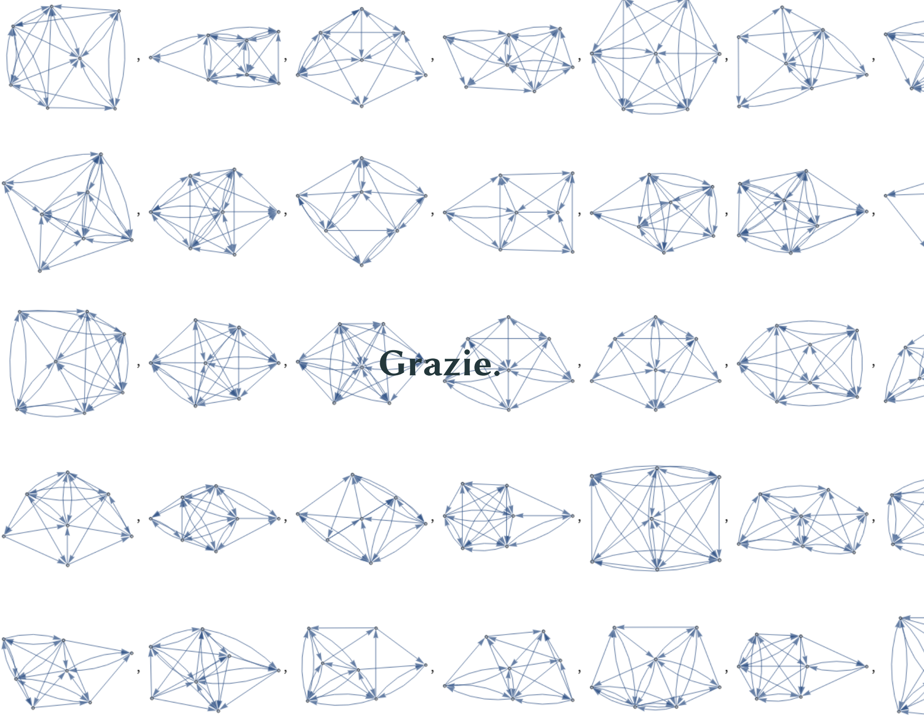
Thank You.





Merci.





Grazie.